VI. *On the Extension and Flexure of Cylindrical and Spherical Thin Elastic Shells.*

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1. The various theories of thin elastic shells which have hitherto been proposed have been discussed by Mr. Love[*] in a recent memoir, and it appears that most, if not all of them, depend upon the assumption that the three stresses which are usually denoted by R, S, T are zero; but, as I have recently pointed out,[†] a very cursory examination of the subject is sufficient to show that this assumption cannot be rigorously true. It can, however, be proved that, when the external surfaces of a plane plate are not subjected to pressure or tangential stress, these stresses depend upon quantities proportional to the square of the thickness, and whenever this is the case they may be treated as zero in calculating the expression for the potential energy due to strain, because they give rise to terms proportional to the fifth power of the thickness, which may be neglected, since it is usually unnecessary to retain powers of the thickness higher than the cube. It will also, in the present paper, be shown by an indirect method that a similar proposition is true in the case of cylindrical and spherical shells, and, therefore, the fundamental hypothesis upon which Mr. Love has based his theory, although unsatisfactory as an assumption, leads to correct results. A general expression for the potential energy due to strain in curvilinear coordinates has also been obtained by Mr. Love, and the equations of motion and the boundary conditions have been deduced therefrom by means of the Principle of Virtual Work, and if this expression and the equations to which it leads were correct, it would be unnecessary to propose a fresh theory of thin shells; but although those portions of Mr. Love's results which depend upon the thickness of the shell are undoubtedly correct, yet, for reasons which will be more fully stated hereafter, I am of opinion that the terms which depend upon the cube of thickness are not strictly accurate, inasmuch as he has omitted to take into account several terms of this order, both in the expression for the potential energy and elsewhere. His preliminary analysis is also of an exceedingly complicated character.

2. Throughout the present paper the notation of Thomson and Tait's "Natural Philosophy" will be employed for stresses and elastic constants, but, for the purpose

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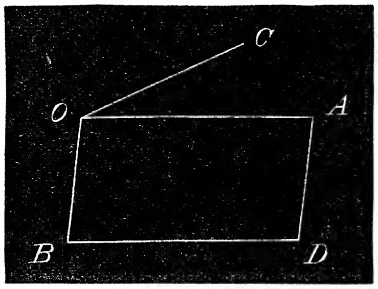
[*] 'Phil. Trans.,' A, 1888, p. 491.

[†] 'London Math. Soc. Proc.,' vol. 21, p. 33.

of facilitating comparison, Mr. LOVE's notation will be employed for strains and directions. It will also be convenient to denote the values of the various quantities involved, at a point P on the middle surface of the shell by *unaccented* letters; and the values of the same quantities at a point P′ on the normal at P, whose distance from P is h', by *accented* letters. The radius of the shell will also be denoted by a, and its thickness by $2h$.

The theory which it is proposed to develop for cylindrical and spherical shells is identical, except in matters of detail, with the theory of plane plates which I recently communicated to the London Mathematical Society,* but for the sake of completeness a short outline will be given.



In the figure let OADB be a small curvilinear rectangle described on the middle surface of the shell, of which the sides are lines of curvature; and let us consider a small element of the shell bounded by the external surfaces, and the four planes passing through the sides of this rectangle, which are perpendicular to the middle surface.

The resultant stresses per unit of length which act upon the element, and which are due to the action of contiguous portions of the shell, are completely specified by the following quantities; viz., across the section AD,

T_1 = a tension across AD parallel to OA,
M_2 = a tangential shearing stress along AD,
N_2 = a normal shearing stress parallel to OC,
G_2 = a flexural couple from C to A, whose axis is parallel to AD,
H_1 = a torsional couple from B to C, whose axis is parallel to OA.

Similarly the resultant stresses per unit of length which act across the section BD are,

T_2 = a tension across BD parallel to OB,
M_1 = a tangential shearing stress along BD,
N_1 = a normal shearing stress parallel to OC,
G_1 = a flexural couple from B to C, whose axis is parallel to BD,
H_2 = a torsional couple from C to A, whose axis is parallel to OB.

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* 'London Math. Soc. Proc.,' vol. 21, p. 33.

If the edges AD, BD were of finite length, there would also be a couple whose axis is parallel to the normal, but since this couple is proportional to the cube of the edge, it vanishes in comparison with the other stresses when the rectangle OADB is indefinitely diminished.

We shall denote the components of the bodily forces per unit of mass in the directions OA, OB, OC by X, Y, Z; but for reasons which will be more fully explained hereafter, we shall suppose that these forces arise solely from external causes, such as gravity and the like. All forces arising from pressures or tangential stresses applied to the surface of the shell will be expressly excluded.

The first step is to write down the equations of motion of an element of the shell in terms of the sectional stresses,* which can be done by the usual methods; we shall thus obtain six equations, three of which are formed by resolving the forces parallel to OA, OB, OC, and three more by taking moments about these lines.

These equations will not, however, enable us to solve any statical or dynamical problems; in order to do this we require the equations of motion in terms of the displacement of a point on the middle surface and their space variations with respect to the coordinates of that point.

3. The values at P′ of all the quantities with which we are concerned are functions of the position of P′, and are, therefore, functions of (r, z, ϕ) or (r, θ, ϕ), according as the shell is cylindrical or spherical. If, therefore, $\mathfrak{Q}'$ be the value of any such quantity at P′, and $\mathfrak{Q}$ the value of the same quantity at the point P, which is the projection of P′ on the middle surface, it follows that

$$\begin{aligned} \mathfrak{Q}' &= \mathrm{F}(r) = \mathrm{F}(a + h') \\ &= \mathfrak{Q} + h'\left(\frac{d\mathfrak{Q}}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2\mathfrak{Q}}{dr^2}\right) + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1) \end{aligned}$$

by Taylor's theorem, where the brackets are employed, as will be done throughout this paper, to denote the values of the differential coefficients at the middle surface where $r = a$.

Objections have been raised by Saint-Venant and endorsed by Mr. Love, to the supposition that the first few terms of the expansion by Taylor's theorem of the quantities involved may be taken as a sufficient approximation. If, however, this objection were valid, it would appear to me to upset the greater part of most physical investigations; inasmuch as it is always assumed as a general principle, that when a quantity is known to be a function of the position of a point P, its value at a neighbouring point P′ may be obtained by Taylor's theorem, unless some physical discontinuity exists in passing from P to P′. If, therefore, we put $\mathfrak{Q} = \mathrm{R}$, we may write

$$\mathrm{R}' = \mathrm{A} + \mathrm{A}_1 h' + \tfrac{1}{2}\mathrm{A}_2 h'^2 + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where the A's are functions of the position of P and also of the thickness of the shell.

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* See Besant "On the Equilibrium of a Bent Lamina," 'Quart. Journ. Math.,' vol. 4, p. 12.

A question which is of fundamental importance in the theory now arises, as to the way in which the A's depend upon h.

If R′ were of the order of the square of the thickness, it is evident that A and A_1 could not contain any powers of h lower than the second and first respectively, whilst A_2 could not contain any negative power of h. The A's are entirely unknown quantities, and as there appears to be no possibility of determining them by an *a priori* method, it seems hopeless to attempt to construct any theory of thin shells without the aid of some assumption which will enable us to get rid of them. *If, however, we assume, as has been practically done by previous writers, that, when the surfaces of the shell are not subjected to any surface forces such as pressures or tangential stresses,* R′ *and also* S′ *and* T′, *so far as they depend on* h *and* h', *are capable of being expressed in the form*

$$u_2 + u_3 + \ldots u_n + \ldots$$

where u_n *is a homogeneous n-tic function of* h *and* h', *the problem can be completely solved without attempting to determine by any* a priori *method the values of any unknown quantities, and upon this fundamental hypothesis the theory of the present paper will be based.*

There is some direct evidence of the truth of this hypothesis. In the case of a plane plate of infinite extent, it can be proved to be true by means of the general equations of motion of an elastic solid;[*] and for the purpose of testing the hypothesis in the case of a curved shell, I have recently investigated to a second approximation, so as to obtain the term in h^2, the period of the radial vibrations of an indefinitely long cylindrical shell, by means of the general equations, and also by means of the theory of thin shells, and both results agree.[†] But far the most conclusive evidence in favour of the truth of the hypothesis is furnished by the results to which it leads; and I have, therefore, conducted the following investigation in such a manner as to furnish a test of the correctness of the final results, and consequently of the fundamental hypothesis by means of which they are deduced.

Having obtained the equations of motion of a cylindrical and a spherical shell in terms of the sectional stresses, all these stresses are then calculated by a direct method, with the exception of the tensions T_1, T_2, which cannot be calculated directly, since they involve the unknown quantities A and A_2. After that the potential energy and the other constitutents of the variational equation are calculated, and the variation worked out by the usual methods. The final result, as is always the case in such investigations, consists of a line integral and a surface integral, the former of which determines the values of the sectional stresses in terms of the displacements, and the latter of which determines in the same form the three equations of motion. Now, if the work and the fundamental hypothesis upon which the theory is based are correct, the variational equation will give the correct values of the tensions T_1, T_2, which are unknown, and will also reproduce the values of the other stresses which have been obtained directly. This is the first test. The second test is furnished by the consideration that, if we substitute the values of the sectional stresses which we have obtained from the variational equation, in the first three of our original equations of motion in terms of these stresses, we ought to reproduce the equations of motion in terms of the displacements, which have been obtained from the variational equation. This is found to be the case both when the shell is cylindrical and when it is spherical; and I therefore think that the fundamental hypothesis is sufficiently established. Having obtained the values of the sectional stresses, the boundary conditions can be deduced by means of STOKES' theorem, which enables us to prove that it is possible to apply a certain distribution of stress to the edge of a thin shell, without producing any alteration in the potential energy due to strain.

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[*] Lord RAYLEIGH, 'London Math. Soc. Proc.,' vol. 20, p. 225; see also vol. 21, p. 33.

[†] 'London Math. Soc. Proc.,' vol. 21, p. 53.

The fundamental hypothesis that R', S', T' may be treated as zero is not true when the surfaces of the shell are subjected to external pressures or tangential stresses; for if the convex and concave surfaces of the shell were subjected to pressures Π_1, Π_2, the value of R' as we pass through the substance of the shell from its exterior to its interior surface, must vary from $-\Pi_1$ to $-\Pi_2$, and consequently (excepting in very special cases) R will contain a term independent of the thickness. Hence the theory developed in the present paper is not applicable to problems relating to the collapse of boiler flues, or to the communication of the vibrations of a vibrating body to the atmosphere. In order to obtain a theory which would enable such questions to be mathematically investigated, it would be necessary to find the values of the additional terms in the variational equation of motion, which depend upon the external pressures; and this is a problem which awaits solution.

It will be convenient briefly to state the notation employed.

In the case of a cylindrical shell, OA is measured along a generating line, and OB along a circular section. In the case of a spherical shell, OA is measured along a meridian, and OB along a parallel of latitude.

The three extensional strains along OA, OB, OC are denoted by σ_1, σ_2, σ_3; and the three shearing strains about those lines by ϖ_1, ϖ_2, ϖ_3. We shall also use the letters λ, μ, p; λ', μ', p' to denote the first and second differential coefficients of σ_1, σ_2, ϖ_3 with respect to r, when $r = a$. We shall also write

$$\begin{aligned} E &= (m-n)/(m+n), & K &= \sigma_1 + \sigma_2, \\ \mathfrak{A} &= \sigma_1 + E(\sigma_1 + \sigma_2), & \mathfrak{B} &= \sigma_2 + E(\sigma_1 + \sigma_2), \\ \mathfrak{E} &= \lambda + E(\lambda + \mu), & \mathfrak{F} &= \mu + E(\lambda + \mu), \\ \mathfrak{E}' &= \lambda' + E(\lambda' + \mu'), & \mathfrak{F}' &= \mu' + E(\lambda' + \mu'). \end{aligned}$$

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4. Before we can obtain the equations of motion or the potential energy, it will be necessary to ascertain the values of the first and second differential coefficients of the displacements with respect to r when $r = a$. We shall, therefore, proceed to calculate these quantities.

Putting λ, μ ; λ', μ' for the values of $(d\sigma_1/dr)$, $(d\sigma_2/dr)$; $(d^2\sigma_1/dr^2)$, $(d^2\sigma_2/dr^2)$ when $r = a$, we have

$$\begin{aligned} \mathrm{R}' &= (m+n)\sigma'_3 + (m-n)(\sigma'_1 + \sigma'_2) \\ &= (m+n)\sigma_3 + (m-n)(\sigma_1 + \sigma_2) \\ &\quad + \left\{(m+n)\left(\frac{d\sigma_3}{dr}\right) + (m-n)(\lambda + \mu)\right\}h' \\ &\quad + \tfrac{1}{2}\left\{(m+n)\left(\frac{d^2\sigma_3}{dr^2}\right) + (m-n)(\lambda' + \mu')\right\}h'^2 + \ldots \qquad (3). \end{aligned}$$

But from (2),

$$\mathrm{R}' = \mathrm{A} + \mathrm{A}_1 h' + \tfrac{1}{2}\mathrm{A}_2 h'^2 + \ldots \qquad (4),$$

whence

$$\left.\begin{aligned} \mathrm{A} &= (m+n)\sigma_3 + (m-n)(\sigma_1 + \sigma_2) \\ \mathrm{A}_1 &= (m+n)\left(\frac{d\sigma_3}{dr}\right) + (m-n)(\lambda + \mu) \\ \mathrm{A}_2 &= (m+n)\left(\frac{d^2\sigma}{dr^2}\right) + (m-n)(\lambda' + \mu') \end{aligned}\right\} \qquad (5),$$

where A, A_1 do not contain any lower powers of h, than h^2 and h respectively, and A_2 does not contain any negative powers of h.

If u', v', w' be the component displacements of any point of the substance of the shell in the direction z, ϕ, r, the equations connecting the displacements and strains are

$$\left.\begin{aligned} \sigma'_1 &= \frac{du'}{dz} \\ \sigma'_2 &= \frac{1}{r}\left(\frac{dv'}{d\phi} + w'\right) \\ \sigma'_3 &= \frac{dw'}{dr} \\ \varpi'_1 &= \frac{dv'}{dr} - \frac{v'}{r} + \frac{1}{r}\frac{dw'}{d\phi} \\ \varpi'_2 &= \frac{dw'}{dz} + \frac{du'}{dr} \\ \varpi'_3 &= \frac{1}{r}\frac{du'}{d\phi} + \frac{dv'}{dz} \end{aligned}\right\} \qquad (6),$$

whence if

$$\mathrm{E} = \frac{m-n}{m+n}, \qquad \mathrm{K} = \sigma_1 + \sigma_2 \qquad (7),$$

we obtain

$$\left(\frac{du}{dr}\right) = \varpi_2 - \frac{dw}{dz}, \quad \left(\frac{dv}{dr}\right) = \varpi_1 + \frac{v}{a} - \frac{1}{a}\frac{dw}{d\phi}, \quad \left(\frac{dw}{dr}\right) = \frac{\mathrm{A}}{m+n} - \mathrm{EK} \qquad (8),$$

and

$$\left.\begin{aligned} \left(\frac{d^2u}{dr^2}\right) &= \left(\frac{d\varpi_2}{dr}\right) - \left(\frac{d^2w}{dz\,dr}\right) = \left(\frac{d\varpi_2}{dr}\right) - \frac{1}{m+n}\frac{d\mathrm{A}}{dz} + \mathrm{E}\frac{d\mathrm{K}}{dz} \\ \left(\frac{d^2v}{dr^2}\right) &= \left(\frac{d\varpi_1}{dr}\right) + \frac{\varpi_1}{a} - \frac{1}{a(m+n)}\frac{d\mathrm{A}}{d\phi} + \frac{\mathrm{E}}{a}\frac{d\mathrm{K}}{d\phi} \\ \left(\frac{d^2w}{dr^2}\right) &= \frac{\mathrm{A}_1}{m+n} - \mathrm{E}\,(\lambda + \mu) \end{aligned}\right\} \quad . \quad . \quad . \quad (9).$$

5. We can now obtain the equations of motion in terms of the sectional stresses.

Let $d\mathrm{S}$ be an element of the middle surface whose coordinates are (a, z, ϕ), and $d\mathrm{S}'$ an element of a layer of the shell whose coordinates are $(a + h', z, \phi)$; then $d\mathrm{S}' = (1 + h'/a)\,d\mathrm{S}$. If we consider a small element of volume bounded by the two external surfaces of the shell, and the four planes passing through the sides of $d\mathrm{S}$, which are perpendicular to the middle surface, we obtain by resolving parallel to OA,

$$\frac{d}{dz}(\mathrm{T}_1 a\,\delta\phi)\,\delta z + \frac{d}{d\phi}(\mathrm{M}_1\,\delta z)\,\delta\phi = \rho\,d\mathrm{S}\int_{-h}^{h}(\ddot{u}' - \mathrm{X})\,(1 + h'/a)\,dh' \qquad (10).$$

But

$$u' = u + h'\left(\frac{du}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2u}{dr^2}\right);$$

accordingly if we substitute the values of (du/dr) and (d^2u/dr^2) from (8) and (9) and recollect that all terms which vanish with h may be omitted when multiplied by h^3, the right hand side of (10) becomes

$$\rho\,d\mathrm{S}\left\{2h\,(\ddot{u} - \mathrm{X}) + \tfrac{1}{3}h^3\,\mathrm{E}\frac{d\ddot{\mathrm{K}}}{dz} - \frac{2h^3}{3a}\frac{d\ddot{w}}{dz}\right\}.$$

Resolving parallel to OB, OC, and then taking moments about OA, OB, OC, we shall obtain in a similar way five other equations, which, together with (10), may be written

$$\left.\begin{aligned} \frac{d\mathrm{T}_1}{dz} + \frac{1}{a}\frac{d\mathrm{M}_1}{d\phi} &= \rho\left\{2h\,(\ddot{u} - \mathrm{X}) + \tfrac{1}{3}h^3\mathrm{E}\frac{d\ddot{\mathrm{K}}}{dz} - \frac{2h^3}{3a}\frac{d\ddot{w}}{dz}\right\} \\ \frac{1}{a}\frac{d\mathrm{T}_2}{d\phi} + \frac{\mathrm{N}_1}{a} + \frac{d\mathrm{M}_2}{dz} &= \rho\left\{2h\,(\ddot{v} - \mathrm{Y}) + \frac{h^3}{3a}\mathrm{E}\frac{d\ddot{\mathrm{K}}}{d\phi} + \frac{2h^3}{3a^2}\left(\ddot{v} - \frac{d\ddot{w}}{d\phi}\right)\right\} \\ \frac{d\mathrm{N}_2}{dz} + \frac{1}{a}\frac{d\mathrm{N}_1}{d\phi} - \frac{\mathrm{T}_2}{a} &= \rho\left\{2h\,(\ddot{w} - \mathrm{Z}) - \tfrac{1}{3}h^3\mathrm{E}\,(\ddot{\lambda} + \ddot{\mu}) - \frac{2h^3}{3a}\mathrm{E}\ddot{\mathrm{K}}\right\} \\ \frac{1}{a}\frac{d\mathrm{G}_1}{d\phi} + \frac{d\mathrm{H}_1}{dz} + \mathrm{N}_1 &= \frac{2\rho h^3}{3a}\left(\frac{d\ddot{w}}{d\phi} - 2\ddot{v} + \mathrm{Y}\right) \\ \frac{d\mathrm{G}_2}{dz} + \frac{1}{a}\frac{d\mathrm{H}_2}{d\phi} - \mathrm{N}_2 &= -\tfrac{2}{3}\rho h^3\left(\frac{d\ddot{w}}{dz} - \frac{\ddot{u}}{a} + \frac{\mathrm{X}}{a}\right) \end{aligned}\right\} \qquad (11).$$

$$(\mathrm{M}_2 - \mathrm{M}_1)\,a - \mathrm{H}_2 = 0.$$

These equations will not enable us to solve the problem in hand; in order to do this we require the equations of motion in terms of the displacements, and also the values of the sectional stresses in terms of the same quantities.

6. The values of the couples, and also the values of M_1, M_2, can be obtained by direct calculation; but the values of T_1, T_2 cannot be so obtained, since they involve the quantities Ah and A_2h^3, which are unknown, and which cannot be neglected. We shall, therefore, be compelled to find the expression for the potential energy, and employ the Calculus of Variations.

The following results will, however, be necessary hereafter. If P', Q', R', S', T', U', be the stresses at the point $a + h'$, z, ϕ, we have

$$\begin{aligned} T_1 a\, \delta\phi &= \int_{-h}^{h} P'(a + h')\, \delta\phi\, dh' \\ &= \int_{-h}^{h} \left\{ P + h'\left(\frac{dP}{dr}\right) + \tfrac{1}{2} h'^2 \left(\frac{d^2P}{dr^2}\right) \right\} \left(1 + \frac{h'}{a}\right) a\, \delta\phi\, dh', \end{aligned}$$

whence

$$\left.\begin{aligned} T_1 &= 2hP + \tfrac{1}{3} h^3 \left(\frac{d^2P}{dr^2}\right) + \frac{2h^3}{3a}\left(\frac{dP}{dr}\right) \\ T_2 &= 2hQ + \tfrac{1}{3} h^3 \left(\frac{d^2Q}{dr^2}\right) \\ M_2 &= 2nh\varpi_3 + \tfrac{1}{3} nh^3 \left(\frac{d^2\varpi_3}{dr^2}\right) + \frac{2nh^3}{3a}\left(\frac{d\varpi_3}{dr}\right) \\ M_1 &= 2nh\varpi_3 + \tfrac{1}{3} nh^3 \left(\frac{d^2\varpi_3}{dr^2}\right) \\ G_1 &= -\tfrac{2}{3} h^3 \left(\frac{dQ}{dr}\right) \\ G_2 &= \tfrac{2}{3} h^3 \left\{\left(\frac{dP}{dr}\right) + \frac{P}{a}\right\} \\ H_1 &= -\tfrac{2}{3} nh^3 \left\{\left(\frac{d\varpi_3}{dr}\right) + \frac{\varpi_3}{a}\right\} \\ H_2 &= \tfrac{2}{3} nh^3 \left(\frac{d\varpi_3}{dr}\right) \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (12).$$

From the third, fourth, and last of these we see that $(M_2 - M_1)\, a = H_2$, as ought to be the case.

Let

$$\left.\begin{aligned} \mathfrak{A} &= \sigma_1 + E(\sigma_1 + \sigma_2), & \mathfrak{B} &= \sigma_2 + E(\sigma_1 + \sigma_2) \\ \mathfrak{E} &= \lambda + E(\lambda + \mu), & \mathfrak{F} &= \mu + E(\lambda + \mu) \end{aligned}\right\} \quad . \; . \; . \; . \quad (13).$$

Then, in the terms multiplied by h^3, we may put

$$P = 2n\mathfrak{A}, \qquad Q = 2n\mathfrak{B},$$

$$\left(\frac{dP}{dr}\right) = 2n\mathfrak{E}, \qquad \left(\frac{dQ}{dr}\right) = 2n\mathfrak{F},$$

whence, if $p = (d\varpi_3/dr)$, the last four of (12) become

$$\left.\begin{array}{ll} G_1 = -\tfrac{4}{3}nh^3\mathfrak{F}, & G_2 = \tfrac{4}{3}nh^3\left(\mathfrak{E} + \dfrac{\mathfrak{A}}{a}\right) \\ H_1 = -\tfrac{2}{3}nh^3\left(p + \dfrac{\varpi_3}{a}\right), & H_2 = \tfrac{2}{3}nh^3p \end{array}\right\} \quad . \quad . \quad . \quad (14).$$

Since the couples are proportional to the cube of the thickness, it follows from the fourth and fifth of (11), that the normal shearing stresses N_1, N_2 are also proportional to the cube of the thickness, and therefore the terms of lowest order in the expressions for the shearing strains ϖ_1', ϖ_2' are quadratic functions of h and h', since such functions when integrated through a section of the shell, give rise to quantities proportional to the cube of the thickness. This is consistent with the fundamental hypothesis.

The next thing is to calculate the values of the quantities λ, μ, p.

From the first and fifth of (6) we obtain

$$\lambda = \left(\frac{d\sigma_1}{dr}\right) = \frac{d\varpi_2}{dz} - \frac{d^2w}{dz^2},$$

and, since the terms in λ are all multiplied by h^3, we may put

$$\lambda = -\frac{d^2w}{dz^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

Similarly from the second and fourth of (6) we obtain

$$\mu = -\frac{1}{a^2}\left(\frac{d^2w}{d\phi^2} + w\right) - \frac{E}{a}(\sigma_1 + \sigma_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

Lastly,

$$p = \left(\frac{d\varpi_3}{dr}\right) = \frac{1}{a}\left(\frac{d^2u}{dr\,d\phi}\right) + \left(\frac{dv}{dr\,dz}\right) - \frac{1}{a^2}\frac{du}{d\phi},$$

or

$$p = -\frac{2}{a}\frac{d^2w}{dz\,d\phi} + \frac{1}{a}\frac{dv}{dz} - \frac{1}{a^2}\frac{du}{d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

We have, therefore, completely determined the values of the couples in terms of known quantities.

We shall also require the values of $(d^2\sigma_1/dr^2)$, $(d^2\sigma_2/dr')$, $(d^2\varpi_3/dr')$, the first two of which we have denoted by λ', μ'; and the last of which we shall denote by p'. The values of these quantities can, by a similar process, be shown to be

$$\left.\begin{array}{l} \lambda' = E\dfrac{d^2K}{dz^2} \\ \mu' = -\dfrac{2\mu}{a} + \dfrac{E}{a^2}\dfrac{d^2K}{d\phi^2} - \dfrac{E}{a}(\lambda + \mu) \\ p' = -\dfrac{p}{a} + \dfrac{\varpi_3}{a^2} + \dfrac{2E}{a}\dfrac{d^2K}{dz\,d\phi} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

Equation (17) and the last of (18), combined with the third and fourth of (12), determine the values of M_1, M_2.

It will be desirable to point out at this stage of the investigation, that we have obtained materials for the complete solution of any problem in which T_1 and u are zero, and none of the quantities are functions of z. The boundary conditions at a free edge will be discussed in § 11, and the reader who does not wish to be troubled with the long analytical process of finding the potential energy and working out the variational equation of motion, may pass at once to § 10, and the following sections where certain problems of a fairly simple kind are discussed.

7. We must now find the potential energy due to strain.

By the ordinary formula, the potential energy of a portion of the shell is

$$W = \tfrac{1}{2}\iiint_{-h}^{h} [(m+n)\Delta'^2$$
$$+ n\{\varpi_1'^2 + \varpi_2'^2 + \varpi_3'^2 - 4(\sigma_1'\sigma_2' + \sigma_2'\sigma_3' + \sigma_3'\sigma_1')\}](1 + h'/a)\,dh'dS \quad (19),$$

where the integration with respect to z and ϕ extends over the middle surface of the portion considered. In evaluating this expression we may at once omit ϖ_1', ϖ_2', for since they are quadratic functions of h and h', they will give rise to terms which are proportional to h^5, which are to be neglected.

Since

$$\Delta' = \Delta + h'\left(\frac{d\Delta}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2\Delta}{dr^2}\right) + \ldots$$

it follows that

$$\tfrac{1}{2}(m+n)\int_{-h}^{h}\Delta'^2\left(1 + \frac{h'}{a}\right)dh'$$
$$= (m+n)\left\{h\Delta^2 + \tfrac{1}{3}h^3\left(\frac{d\Delta}{dr}\right)^2 + \tfrac{1}{3}h^3\Delta\left(\frac{d^2\Delta}{dr^2}\right) + \frac{2h^3}{3a}\Delta\left(\frac{d\Delta}{dr}\right)\right\},$$

from which it is seen that W is expressible in a series of odd powers of h.

From (5) we obtain

$$\Delta = (1 - E)(\sigma_1 + \sigma_2) + \frac{A}{m+n},$$

$$\left(\frac{d\Delta}{dr}\right) = (1 - E)(\lambda + \mu) + \frac{A_1}{m+n},$$

$$\left(\frac{d^2\Delta}{dr^2}\right) = (1 - E)(\lambda' + \mu') + \frac{A_2}{m+n},$$

and, therefore, the portion of W per unit of area of the middle surface, which depends upon Δ', is

$$\frac{4n^2}{m+n}\Big\{h\,(\sigma_1+\sigma_2+\mathrm{A}/2n)^2+\tfrac{1}{3}h^3(\lambda+\mu+\mathrm{A}_1/2n)^2$$
$$+\tfrac{1}{3}h^3(\sigma_1+\sigma_2+\mathrm{A}/2n)(\lambda'+\mu'+\mathrm{A}_2/2n)+\frac{2h^3}{3a}(\sigma_1+\sigma_2+\mathrm{A}/2n)(\lambda+\mu+\mathrm{A}_1/2n)\Big\} \quad (20)$$

in which in the last three terms we may omit the A's since they are multiplied by h^3.

Again

$$\sigma_1'\sigma_2'\,(1+h'/a)=\sigma_1\sigma_2+\lambda\mu h'^2+\tfrac{1}{2}(\lambda'\sigma_2+\mu'\sigma_1)\,h'^2+(\lambda\sigma_2+\mu\sigma_1)\,h'^2/a+\;.\;\;.$$

whence

$$2n\int_{-h}^{h}\sigma_1'\sigma_2'\,(1+h'/a)\,dh'=4nh\sigma_1\sigma_2+\tfrac{4}{3}nh^3\lambda\mu+\tfrac{2}{3}nh^3\,(\lambda'\sigma_2+\mu'\sigma_1)$$
$$+\frac{4nh^3}{3a}(\lambda\sigma_2+\mu\sigma_1) \qquad (21).$$

Also

$$(\sigma_1'+\sigma_2')\,\sigma_3'\,(1+h'/a)=(\sigma_1+\sigma_2)\,\sigma_3+h'^2(\lambda+\mu)\left(\frac{d\sigma_3}{dr}\right)+\tfrac{1}{2}h'^2(\lambda'+\mu')\,\sigma_3$$
$$+\tfrac{1}{2}h'^2(\sigma_1+\sigma_2)\left(\frac{d^2\sigma_3}{dr^2}\right)+\frac{h'^2}{a}\left\{(\lambda+\mu)\,\sigma_3+(\sigma_1+\sigma_2)\left(\frac{d\sigma_3}{dr}\right)\right\},$$

whence

$$2n\int_{-h}^{h}(\sigma_1'+\sigma_2')\,\sigma_3'\,(1+h'/a)\,dh'=4nh\,(\sigma_1+\sigma_2)\left\{\frac{\mathrm{A}}{m+n}-\mathrm{E}\,(\sigma_1+\sigma_2)\right\}$$
$$-\tfrac{4}{3}nh^3\mathrm{E}(\lambda+\mu)^2-\tfrac{4}{3}nh^3\mathrm{E}\,(\sigma_1+\sigma_2)\,(\lambda'+\mu')-\frac{8nh^3}{3a}\mathrm{E}\,(\lambda+\mu)(\sigma_1+\sigma_2) \qquad (22).$$

Lastly

$$\tfrac{1}{2}n\int_{-h}^{h}\varpi_3'^2\,(1+h'/a)\,dh'=nh\varpi_3^2+\tfrac{1}{3}nh^3p^2+\tfrac{1}{3}nh^3\varpi_3p'+\frac{2nh^3}{3a}\varpi_3p \qquad (23).$$

Substituting from (20), (21), (22), (23) in (19), it will be found that the term Ah, which is (or at any rate may be) proportional to h^3, disappears; and thus the value of the potential energy per unit of the area of the middle surface is

$$\begin{aligned}\mathrm{W}=\;&2nh\,\{\sigma_1^2+\sigma_2^2+\mathrm{E}\,(\sigma_1+\sigma_2)^2+\tfrac{1}{2}\varpi^2\}\\&+\tfrac{2}{3}nh^3\,\{\lambda^2+\mu^2+\mathrm{E}\,(\lambda+\mu)^2+\tfrac{1}{2}p^2\}\\&+\tfrac{2}{3}nh^3\,(\mathfrak{A}\lambda'+\mathfrak{B}\mu'+\tfrac{1}{2}\varpi p')\\&+\tfrac{4}{3}\frac{nh^3}{a}\,(\mathfrak{A}\lambda+\mathfrak{B}\mu+\tfrac{1}{2}\varpi p)\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \quad (24),\end{aligned}$$

in which ϖ is written for ϖ_3, the suffix being no longer required.

This is the expression for the potential energy as far as the term involving h^3. The first term depends solely upon the extension of the middle surface; the second term depends principally upon the quantities by which the bending is specified, and the third and fourth consist of the products of the extensions and the quantities which principally depend upon the bending.

8. Having obtained the value of the potential energy, we must in the next place form the variational equation of motion. This equation may symbolically be written

$$\delta \mathrm{W} + \delta \mathfrak{T} = \delta \mathrm{U} + \delta \mathfrak{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25),$$

where $\delta \mathfrak{T}$ is the term which depends upon the time variations of the displacements, $\delta \mathrm{U}$ is the work done by the bodily forces, and $\delta \mathfrak{L}$ represents the work done upon the edges of the portion of the shell considered, in producing the displacements, δu, δv, δw, by the forces arising from the action of contiguous portions of the shell. It, therefore, follows that $\delta \mathfrak{L}$ is a line integral taken round the edge of the portion of the shell which is being considered; and as one of our objects is to calculate the values of the sectional stresses in terms of the displacements by means of (25), it will be convenient to apply the variational equation to a curvilinear rectangle bounded by four lines of curvature.

We must now calculate $\delta \mathfrak{T}$. We have

$$\delta \mathfrak{T} = \rho \iiint_{-h}^{h} (\ddot{u}' \, \delta u' + \ddot{v}' \, \delta v' + \ddot{w}' \, \delta w') \, (1 + h'/a) \, dh' \, d\mathrm{S}.$$

Now

$$\begin{aligned} \int_{-h}^{h} \ddot{u}' \, \delta u' \, (1 + h'/a) \, dh' &= 2h\ddot{u} \, \delta u + \tfrac{2}{3} h^3 \frac{d\ddot{u}}{dr} \frac{d\delta u}{dr} + \tfrac{1}{3} h^3 \left(\ddot{u} \frac{d^2 \delta u}{dr^2} + \frac{d^2 \ddot{u}}{dr^2} \delta u \right) \\ &\qquad + \frac{2h^3}{3a} \left(\ddot{u} \frac{d\delta u}{dr} + \frac{d\ddot{u}}{dr} \delta u \right) \\ &= 2h\ddot{u} \, \delta u + \tfrac{2}{3} h^3 \frac{d\ddot{w}}{dz} \frac{d\delta w}{dz} + \tfrac{1}{3} h^3 \mathrm{E} \left(\ddot{u} \frac{d\delta \mathrm{K}}{dz} + \frac{d\ddot{\mathrm{K}}}{dz} \delta u \right) \\ &\qquad - \frac{2h^3}{3a} \left(\ddot{u} \frac{d\delta w}{dz} + \frac{d\ddot{w}}{dz} \delta u \right) \end{aligned}$$

by (8) and (9). Treating the other terms in a similar way, we shall find that the value of $\delta \mathfrak{T}$ is

$$\begin{aligned} \delta \mathfrak{T} = {} & 2\rho h \iint (\ddot{u} \, \delta u + \ddot{v} \, \delta v + \ddot{w} \, \delta w) \, d\mathrm{S} \\ & + \tfrac{2}{3} \rho h^3 \iint \left\{ \frac{d\ddot{w}}{dz} \frac{d\delta w}{dz} + \frac{1}{a^2} \left(\frac{d\ddot{w}}{d\phi} - \ddot{v} \right) \left(\frac{d\delta w}{d\phi} - \delta v \right) + \mathrm{E}^2 \dot{\mathrm{K}} \, \delta \mathrm{K} \right\} d\mathrm{S} \\ & + \tfrac{1}{3} \rho h^3 \mathrm{E} \iint \left\{ \ddot{u} \frac{d\delta \mathrm{K}}{dz} + \frac{\ddot{v}}{a} \frac{d\delta \mathrm{K}}{d\phi} - \ddot{w} \, (\delta\lambda + \delta\mu) + \frac{d\ddot{\mathrm{K}}}{dz} \delta u + \frac{1}{a} \frac{d\ddot{\mathrm{K}}}{d\phi} \delta v - (\dot{\lambda} + \ddot{\mu}) \, \delta w \right\} d\mathrm{S} \\ & - \frac{2\rho h^3}{3a} \iint \left\{ \ddot{u} \frac{d\delta w}{dz} + \frac{d\ddot{w}}{dz} \delta u + \frac{\ddot{v}}{a} \left(\frac{d\delta w}{d\phi} - \delta v \right) + \frac{1}{a} \left(\frac{d\ddot{w}}{d\phi} - \ddot{v} \right) \delta v \right. \\ & \qquad \left. + \mathrm{E} \, (\ddot{w} \, \delta \mathrm{K} + \ddot{\mathrm{K}} \, \delta w) \right\} d\mathrm{S} \quad . \quad . \quad (26). \end{aligned}$$

We must, in the next place, calculate $\delta\mathfrak{L}$. We have

$$\delta\mathfrak{L} = \iint_{-h}^{h} (\mathrm{P}'\,\delta u' + \mathrm{U}'\,\delta v')\,(d + h')\,dh'\,d\phi + \iint_{-h}^{h} (\mathrm{Q}'\,\delta v' + \mathrm{U}'\,\delta u')\,dh'\,dz$$

$$+ \int \mathrm{N}_2 \delta w/a\,d\phi + \int \mathrm{N}_1\,\delta w\,dz \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (27).$$

From the way in which $\delta\mathfrak{T}$ has been calculated, we see from (8), (9), and (12), that

$$\int_{-h}^{h} \mathrm{P}'\,\delta u'\,(1 + h'/a)\,dh' = \mathrm{T}_1\,\delta u + \mathrm{G}_2\,\frac{d\delta u}{dr} + \tfrac{1}{3}\,h^3\mathrm{P}\,\frac{d^2\delta u}{dr^2}$$

$$= \mathrm{T}_1\,\delta u - \mathrm{G}_2\,\frac{d\delta w}{dz} + \tfrac{2}{3}\,nh^3\mathrm{E}\mathfrak{A}\,\frac{d\delta\mathrm{K}}{dz}\,.$$

Treating the other terms in a similar way, we find

$$\delta\mathfrak{L} = \int\Big\{\mathrm{T}_1\,\delta u + \mathrm{M}_2\,\delta v + \mathrm{N}_2\,\delta w - \mathrm{G}_2\,\frac{d\delta w}{dz} + \frac{\mathrm{H}_1}{a}\Big(\frac{d\delta w}{d\phi} - \delta v\Big)$$

$$+ \tfrac{2}{3}\,nh^3\mathrm{E}\mathfrak{A}\,\frac{d\delta\mathrm{K}}{dz} + \frac{nh^3\mathrm{E}\varpi}{3a}\,\frac{d\delta\mathrm{K}}{d\phi}\Big\}\,a\,d\phi$$

$$+ \int\Big\{\mathrm{M}_1\,\delta u + \mathrm{T}_2\,\delta v + \mathrm{N}_1\,\delta w + \frac{\mathrm{G}_1}{a}\Big(\frac{d\delta w}{d\phi} - \delta v\Big) - \mathrm{H}_2\,\frac{d\delta w}{dz}$$

$$+ \frac{2nh^3}{3a}\,\mathrm{E}\mathfrak{B}\,\frac{d\delta\mathrm{K}}{\delta\phi} + \tfrac{1}{3}\,nh^3\mathrm{E}\varpi\,\frac{d\delta\mathrm{K}}{dz}\Big\}\,dz \quad . \; . \quad (28).$$

Lastly, since the shell is supposed to be so thin that X, Y, Z, may be treated as constants during the integration with respect to h',

$$\delta\mathrm{U} = \rho\iiint_{-h}^{h} (\mathrm{X}\,\delta u' + \mathrm{Y}\,\delta v' + \mathrm{Z}\,\delta w')\,(1 + h'/a)\,dh'\,d\mathrm{S}$$

$$= 2\rho\,h\iint(\mathrm{X}\,\delta u + \mathrm{Y}\,\delta v + \mathrm{Z}\,\delta w)\,d\mathrm{S}$$

$$+ \tfrac{1}{3}\rho\,h^3\mathrm{E}\iint\Big\{\mathrm{X}\,\frac{d\delta\mathrm{K}}{dz} + \frac{\mathrm{Y}}{a}\,\frac{d\delta\mathrm{K}}{d\phi} - \mathrm{Z}\,(\delta\lambda + \delta\mu)\Big\}\,d\mathrm{S}$$

$$- \frac{2\rho h^3}{3a}\iint\Big\{\mathrm{X}\,\frac{d\delta w}{dz} + \frac{\mathrm{Y}}{a}\Big(\frac{d\delta w}{d\phi} - \delta v\Big) + \mathrm{ZE}\,\delta\mathrm{K}\Big\}\,d\mathrm{S} \quad . \; . \; . \quad (29).$$

9. We have now obtained all the materials for the complete solution of the problem, and we shall proceed to work out the variation in the ordinary way.

Let us denote the four terms of the expression for W given in (24), when integrated over a curvilinear rectangle bounded by four lines of curvature, by W_1, W_2, W_3, W_4. Then

$$\delta\mathrm{W}_1 = 4nh\iint(\mathfrak{A}\delta\sigma_1 + \mathfrak{B}\delta\sigma_2 + \tfrac{1}{2}\,\varpi\delta\varpi)\,a\,dz\,d\phi.$$

Substituting the values of σ_1, σ_2, ϖ from the first, second, and sixth of (6), and integrating by parts we shall obtain

$$\delta \mathrm{W}_1 = 4nh\int(\mathfrak{A}\delta u + \tfrac{1}{2}\varpi\delta v)\,a\,d\phi + 4nh\int(\tfrac{1}{2}\varpi\delta u + \mathfrak{B}\delta v)\,dz$$

$$- 4nh\iint\left\{\left(\frac{d\mathfrak{A}}{dz} + \frac{1}{2a}\frac{d\varpi}{d\phi}\right)\delta u + \left(\frac{1}{a}\frac{d\mathfrak{B}}{d\phi} + \frac{1}{2}\frac{d\varpi}{dz}\right)\delta v - \frac{\mathfrak{B}}{a}\delta w\right\}a\,dz\,d\phi \quad . \quad . \quad (30).$$

Now $\delta\mathrm{W}_2$, $\delta\mathrm{W}_3$, $\delta\mathrm{W}_4$ depend upon h^3; if therefore we substitute in (25) the value of $\delta\mathrm{W}_1$ from (30), and the portions of $\delta\mathfrak{T}$, $\delta\mathrm{U}$, and $\delta\mathfrak{L}$, which depend upon h, we shall obtain the approximate equations

$$\left.\begin{array}{ll} \mathrm{T}_1 = 4nh\,\mathfrak{A}, & \mathrm{T}_2 = 4nh\,\mathfrak{B} \\ \mathrm{M}_1 = \mathrm{M}_2 = 2nh\varpi & \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

and

$$\left.\begin{array}{l} \rho\ddot{u} = 2n\left(\dfrac{d\mathfrak{A}}{dz} + \dfrac{1}{2a}\dfrac{d\varpi}{d\phi}\right) + \rho\mathrm{X} \\ \rho\ddot{v} = 2n\left(\dfrac{1}{a}\dfrac{d\mathfrak{B}}{d\phi} + \dfrac{1}{2}\dfrac{d\varpi}{dz}\right) + \rho\mathrm{Y} \\ \rho\ddot{w} = -\,2n\,\mathfrak{B}/a + \rho\mathrm{Z} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

These equations are the same as those obtained by Mr. LOVE,* and which are employed by him in discussing the vibrations of a cylindrical shell. The complete equations giving $\rho\ddot{u}$, $\rho\ddot{v}$, $\rho\ddot{w}$ in terms of the displacements and their space variations contain certain additional terms involving h^2 (since the common factor h disappears) which it is our object to determine; but, since we do not retain terms higher than h^3, we may, if convenient, substitute the above approximate values in all terms of (25) which are multiplied by h^3.

Again

$$\delta\mathrm{W}_2 = \tfrac{4}{3}\,nh^3\iint(\mathfrak{E}\,\delta\lambda + \mathfrak{F}\,\delta\mu + \tfrac{1}{2}p\,\delta p)\,a\,dz\,d\phi.$$

Substituting the values of λ, μ, p from (15), (16), and (17), we obtain

$$\iint\mathfrak{E}\,\delta\lambda\,dz\,d\phi = -\iint\mathfrak{E}\,\frac{d^2\,\delta w}{dz^2}\,dz\,d\phi$$

$$= \int\left(\frac{d\mathfrak{E}}{dz}\,\delta w - \mathfrak{E}\,\frac{d\,\delta w}{dz}\right)d\phi - \iint\frac{d^2\,\mathfrak{E}}{dz^2}\,\delta w\,dz\,d\phi \quad . \quad . \quad . \quad . \quad . \quad (33),$$

also

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* 'Phil. Trans.,' A., 1888, pp. 538 and 540. Equations (32) correspond to LOVE's equations (86), (87), and (88); and (31) to (101).

$$\begin{aligned}\iint\mathfrak{F}\,\delta\mu\,dz\,d\phi &= -\frac{1}{a^2}\iint\mathfrak{F}\left\{\frac{d^2\,\delta w}{d\phi^2}+\delta w+\mathrm{E}\left(a\frac{d\,\delta u}{dz}+\frac{d\,\delta v}{d\phi}+\delta w\right)\right\}dz\,d\phi\\ &= -\frac{\mathrm{E}}{a}\int\mathfrak{F}\,\delta u\,d\phi-\frac{1}{a^2}\int\left(\mathrm{E}\mathfrak{F}\,\delta v-\frac{d\mathfrak{F}}{d\phi}\,\delta w+\mathfrak{F}\,\frac{d\delta w}{d\phi}\right)dz\\ &\quad+\frac{1}{a^2}\iint\left\{\mathrm{E}a\,\frac{d\mathfrak{F}}{dz}\,\delta u+\mathrm{E}\,\frac{d\mathfrak{F}}{d\phi}\,\delta v-\left(\frac{d^2\mathfrak{F}}{d\phi^2}+\mathfrak{F}+\mathrm{E}\mathfrak{F}\right)\delta w\right\}dz\,d\phi \qquad (34),\end{aligned}$$

and

$$\begin{aligned}\tfrac{1}{2}\iint p\,\delta p\,dz\,d\phi &= \frac{1}{2a^2}\iint p\left(a\,\frac{d\delta v}{dz}-\frac{d\delta u}{d\phi}-2a\,\frac{d^2\delta w}{dz\,d\phi}\right)dz\,d\phi\\ &= \frac{1}{2a}\int p\,\delta v\,d\phi-\frac{1}{2a^2}\int p\,\delta u\,dz+\frac{1}{2a^2}\iint\left(\frac{dp}{d\phi}\,\delta u-a\,\frac{dp}{dz}\,\delta v\right)dz\,d\phi\\ &\quad-\frac{1}{a}\iint p\,\frac{d^2\delta w}{dz\,d\phi}\,dz\,d\phi.\end{aligned}$$

The last integral can be evaluated in two different ways, according as we integrate, first, with respect to ϕ, and secondly, with respect to z; or first with respect to z, and secondly with respect to ϕ. The proper way to deal with such a term is, to evaluate the integral in both ways, and then multiply the two values by β and $1-\beta$, and add, where β is a quantity which must be determined from the conditions of the problem in hand. We shall thus find that the value of β is $\frac{1}{2}$; we therefore obtain

$$\begin{aligned}\tfrac{1}{2}\iint p\,\delta p\,dz\,d\phi &= \frac{1}{2a}\int\left(p\,\delta v+\frac{dp}{d\phi}\,\delta w-p\,\frac{d\delta w}{d\phi}\right)d\phi\\ &\quad-\frac{1}{2a^2}\int\left(p\,\delta u-a\,\frac{dp}{dz}\,\delta w+ap\,\frac{d\delta w}{dz}\right)dz\\ &\quad+\frac{1}{2a^2}\iint\left(\frac{dp}{d\phi}\,\delta u-a\,\frac{dp}{dz}\,\delta v-2a\,\frac{d^2p}{dz\,d\phi}\,\delta w\right)dz\,d\phi \quad .\;.\;.\;. \quad (35).\end{aligned}$$

Collecting all the terms together from (33), (34), and (35), we finally obtain

$$\begin{aligned}\delta\mathrm{W}_2 &= \tfrac{4}{3}\,nh^3\int\left\{-\frac{\mathrm{E}\mathfrak{F}}{a}\,\delta u+\left(\frac{d\mathfrak{E}}{dz}+\frac{1}{2a}\frac{dp}{d\phi}\right)\delta w-\mathfrak{E}\,\frac{d\delta w}{dz}-\frac{p}{2a}\left(\frac{d\delta w}{d\phi}-\delta v\right)\right\}a\,d\phi\\ &\quad+\tfrac{4}{3}\,nh^3\int\left\{-\frac{p}{2a}\,\delta u-a\,\frac{(1+\mathrm{E})\,\mathfrak{F}}{a}\,\delta v+\left(\frac{1}{a}\frac{d\mathfrak{F}}{d\phi}+\tfrac{1}{2}\,\frac{dp}{dz}\right)\delta w\right.\\ &\qquad\qquad\left.-\frac{\mathfrak{F}}{a}\left(\frac{d\delta w}{d\phi}-\delta v\right)-\tfrac{1}{2}\,p\,\frac{d\delta w}{dz}\right\}dz\\ &\quad+\frac{4nh^3}{3a}\iint\left[\left(\mathrm{E}\,\frac{d\mathfrak{F}}{dz}+\frac{1}{2a}\frac{dp}{d\phi}\right)\delta u+\left(\frac{\mathrm{E}}{a}\,\frac{d\mathfrak{F}}{d\phi}-\tfrac{1}{2}\,\frac{dp}{dz}\right)\delta v\right.\\ &\qquad\left.-\left\{a\,\frac{d\,\mathfrak{E}}{dz^2}+\frac{1}{a}\left(\frac{d^2\mathfrak{F}}{d\phi^2}+\mathfrak{F}+\mathrm{E}\mathfrak{F}\right)+\frac{d^2p}{dz\,d\phi}\right\}\delta w\right]a\,dz\,d\phi \qquad (36).\end{aligned}$$

Let

$$\mathfrak{E}' = \lambda' + \mathrm{E}(\lambda' + \mu'), \qquad \mathfrak{F}' = \mu' + \mathrm{E}(\lambda' + \mu') \quad . \quad . \quad . \quad . \quad (37),$$

then

$$\delta \mathrm{W}_3 = \tfrac{2}{3}\, nh^3 \iint (\mathfrak{E}'\delta\sigma_1 + \mathfrak{F}'\delta\sigma_2 + \tfrac{1}{2}\, p'\delta\varpi + \mathfrak{A}\,\delta\lambda' + \mathfrak{B}\,\delta\mu' + \tfrac{1}{2}\,\varpi\delta p')\, d\mathrm{S}\,;$$

from this result, together with (24), it is seen that $\delta\mathrm{W}_3$ and $\delta\mathrm{W}_4$ each consist of two parts, which may be denoted by $\delta\mathrm{W}_3{}'$, $\delta\mathrm{W}_3{}''$ and $\delta\mathrm{W}_4{}'$, $\delta\mathrm{W}_4{}''$ respectively. The values of $\delta\mathrm{W}_3{}'$ and $\delta\mathrm{W}_4{}'$ may at once be written down from (30), by changing $\mathfrak{A}$, $\mathfrak{B}$, ϖ into $\mathfrak{E}'$, $\mathfrak{F}'$, p' and $\mathfrak{E}$, $\mathfrak{F}$, p respectively, and by altering the coefficients from $4nh$ into $\frac{2}{3}nh^3$ and $4nh^3/3a$ respectively. With regard to $\delta\mathrm{W}_3{}''$ we have

$$\delta\mathrm{W}_3{}'' = \tfrac{2}{3}\, nh^3 \iint (\mathfrak{A}\,\delta\lambda' + \mathfrak{B}\,\delta\mu' + \tfrac{1}{2}\,\varpi\delta p')\, d\mathrm{S}.$$

Substituting the value of λ' from (18) and integrating once by parts, we obtain

$$\begin{aligned}\iint \mathfrak{A}\,\delta\lambda' d\mathrm{S} &= \iint \mathrm{E}\,\mathfrak{A}\,\frac{d^2\delta\mathrm{K}}{dz^2}\, d\mathrm{S} \\ &= \mathrm{E}\int \mathfrak{A}\,\frac{d\delta\mathrm{K}}{dz}\, a d\phi - \mathrm{E}\iint \frac{d\mathfrak{A}}{dz}\,\frac{d\delta\mathrm{K}}{dz}\, d\mathrm{S}.\end{aligned}$$

Treating the other terms in a similar way, we shall finally obtain

$$\begin{aligned}\delta\mathrm{W}_3{}'' = &\int \left(\tfrac{2}{3}\, nh^3 \mathrm{E}\mathfrak{A}\,\frac{d\delta\mathrm{K}}{dz} + \frac{nh^3}{3a}\,\mathrm{E}\varpi\,\frac{d\delta\mathrm{K}}{d\phi}\right) a d\phi + \int \left(\frac{2nh^3}{3a}\,\mathrm{E}\mathfrak{B}\,\frac{d\delta\mathrm{K}}{d\phi} + \tfrac{1}{3}\, nh^3 \mathrm{E}\varpi\,\frac{d\delta\mathrm{K}}{dz}\right) dz \\ &- \tfrac{2}{3}\, nh^3 \iint \left\{\mathrm{E}\left(\frac{d\mathfrak{A}}{dz} + \frac{1}{2a}\,\frac{d\varpi}{d\phi}\right)\frac{d\delta\mathrm{K}}{dz} + \frac{\mathrm{E}}{a}\left(\frac{1}{a}\,\frac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\,\frac{d\varpi}{dz}\right)\frac{d\delta\mathrm{K}}{d\phi} - \frac{\mathrm{E}\mathfrak{B}}{a}\,(\delta\lambda + \delta\mu)\right\} d\mathrm{S} \\ &+ \tfrac{2}{3}\,\frac{nh^3}{a} \iint \left\{-2\mathfrak{B}\,\delta\mu + \frac{\varpi}{2a}\,(\delta\varpi - a\delta p)\right\} d\mathrm{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).\end{aligned}$$

If in the first surface integral in this equation, we substitute the approximate values of the coefficients of $d\delta\mathrm{K}/dz$, &c., from (32), which we may do, since this integral is multiplied by h^3, and then substitute the values of $\delta\mathrm{W}_3{}''$, $\delta\mathfrak{T}$, $\delta\mathrm{U}$, and $\delta\mathfrak{L}$ in (25), it will be found that all the terms involving $d\delta\mathrm{K}/dz$, $d\delta\mathrm{K}/d\phi$, and $\delta\lambda + \delta\mu$ cut out; we are, therefore, no longer concerned with them, and the value of $\delta\mathrm{W}_3{}''$ reduces to the last line. On this understanding we may, therefore, write

$$\begin{aligned}\delta\mathrm{W}_3{}'' + \delta\mathrm{W}_4{}'' &= \frac{4nh^3}{3a} \iint \left(\mathfrak{A}\,\delta\lambda + \frac{\varpi}{4a}\,\delta\varpi + \tfrac{1}{4}\,\varpi\delta p\right) d\mathrm{S} \\ &= -\frac{4nh^3}{3a}\int \left\{\mathfrak{A}\,\frac{d\delta w}{dz} + \frac{\varpi}{2a}\left(\frac{d\delta w}{d\phi} - \delta v\right)\right\} a d\phi \\ &\quad + \frac{4nh^3}{3a}\iint \left\{\frac{d\mathfrak{A}}{dz}\,\frac{d\delta w}{dz} + \frac{1}{2a}\,\frac{d\varpi}{dz}\left(\frac{d\delta w}{d\phi} - \delta v\right)\right\} d\mathrm{S} \quad . \quad . \quad . \quad (39).\end{aligned}$$

We are now in a position to test the correctness of some of our work, for picking out the terms involving $d\delta w/d\phi - \delta v$, $d\delta w/dz$ in the line integrals in (36) and (39), and equating them to the corresponding terms in the value of $\delta\mathfrak{L}$ which is given by (28), we see that we have reproduced the values of the couples, which we have already obtained in equation (14). We may therefore leave the couple terms out henceforth.

Collecting all our results from (26), (28), (29), and (39) the variational equation becomes

$$\begin{aligned}\delta \mathrm{W}_1 + \delta \mathrm{W}_2 + \delta \mathrm{W}_3{}' + \delta \mathrm{W}_4{}' &+ \tfrac{4}{3}\frac{nh^3}{a}\iint\left\{\frac{d\mathfrak{A}}{dz}\frac{d\delta w}{dz} + \frac{1}{2a}\frac{d\varpi}{dz}\left(\frac{d\delta w}{d\phi} - \delta v\right)\right\}d\mathrm{S}\\
&+ 2\rho h\iint(\ddot{u}\delta u + \ddot{v}\delta v + \ddot{w}\delta w)\,d\mathrm{S}\\
&+ \tfrac{2}{3}\rho h^3\iint\left\{\left(\frac{d\ddot{w}}{dz} - \frac{\ddot{u}}{a}\right)\frac{d\delta w}{dz} + \frac{1}{a^2}\left(\frac{d\ddot{w}}{d\phi} - 2\ddot{v}\right)\left(\frac{d\delta w}{d\phi} - \delta v\right) + \mathrm{E}\,(\mathrm{E}\ddot{\mathrm{K}} - \ddot{w}/a)\,\delta \mathrm{K}\right\}d\mathrm{S}\\
&+ \tfrac{1}{3}\rho h^3\mathrm{E}\iint\left\{\frac{d\ddot{\mathrm{K}}}{dz}\delta u + \frac{1}{a}\frac{d\ddot{\mathrm{K}}}{d\phi}\delta v - (\ddot{\lambda} + \ddot{\mu})\,\delta w\right\}d\mathrm{S}\\
&- \frac{2\rho h^3}{3a}\iint\left\{\frac{d\ddot{w}}{dz}\delta u + \frac{1}{a}\left(\frac{d\ddot{w}}{d\phi} - \ddot{v}\right)\delta v + \mathrm{E}\ddot{\mathrm{K}}\,\delta w\right\}d\mathrm{S}\\
= 2\rho h\iint(\mathrm{X}\,\delta u + \mathrm{Y}\,\delta v + \mathrm{Z}\,\delta w)\,d\mathrm{S} &- \frac{2\rho h^3}{3a}\iint\left\{\mathrm{X}\frac{d\delta w}{dz} + \frac{\mathrm{Y}}{a}\left(\frac{d\delta w}{d\phi} - \delta v\right) + \mathrm{ZE}\,\delta \mathrm{K}\right\}d\mathrm{S}\\
&+ \int(\mathrm{T}_1\,\delta u + \mathrm{M}_2\,\delta v + \mathrm{N}_2\,\delta w)\,a\,d\phi + \int(\mathrm{M}_1\,\delta u + \mathrm{T}_2\,\delta v + \mathrm{N}_1\,\delta w)\,dz \quad . \quad . \quad (40)\end{aligned}$$

where the values of $\delta \mathrm{W}_1$, $\delta \mathrm{W}_2$ are given by (30) and (36), and the values of $\delta \mathrm{W}_3{}'$, $\delta \mathrm{W}_4{}'$ are obtained from (30) by changing certain letters as we have explained above.

We have now got rid of all the terms involving the second differential coefficients of δu, δv, δw, and all that now remains to be done is to integrate by parts the terms which involve the first differential coefficients. Putting

$$\alpha = \frac{2n}{\rho a}\frac{d\mathfrak{A}}{dz} + \frac{d\ddot{w}}{dz} - \frac{\ddot{u}}{a} + \frac{\mathrm{X}}{a},$$

$$\beta = \frac{n}{\rho a}\frac{d\varpi}{dz} + \frac{1}{a}\left(\frac{d\ddot{w}}{d\phi} - 2\ddot{v}\right) + \frac{\mathrm{Y}}{a},$$

$$\gamma = \mathrm{E}\,(\mathrm{E}\ddot{\mathrm{K}} - \ddot{w}/a + \mathrm{Z}/a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41),$$

we have

$$
\begin{aligned}
\tfrac{2}{3}\rho h^3 \iint \left(\alpha \frac{d\,\delta w}{dz} + \frac{\beta}{a}\frac{d\,\delta w}{d\phi} + \gamma\,\delta\mathrm{K}\right) d\mathrm{S} \\
= \tfrac{2}{3}\rho h^3 \int (\gamma\,\delta u + \alpha\,\delta w)\, a\, d\phi + \tfrac{2}{3}\rho h^3 \int (\gamma\,\delta v + \beta\,\delta w)\, dz \\
- \tfrac{2}{3}\rho h^3 \iint \left\{\frac{d\gamma}{dz}\,\delta u + \frac{1}{a}\frac{d\gamma}{d\phi}\,\delta v + \left(\frac{da}{dz} + \frac{1}{a}\frac{d\beta}{d\phi} - \frac{\gamma}{a}\right)\delta w\right\} d\mathrm{S} \quad . \quad . \quad . \quad (42).
\end{aligned}
$$

Substituting the values of $\delta\mathrm{W}_1$, $\delta\mathrm{W}_2$, $\delta\mathrm{W}_3'$, $\delta\mathrm{W}_4'$, and the right hand side of (42) in (40), and picking out the line integral terms, we obtain the following equations for the sectional stresses, viz.,

$$
\left.
\begin{aligned}
\mathrm{T}_1 &= 4nh\mathfrak{A} - \frac{4nh^3}{3a}\mathrm{E}\mathfrak{F} + \tfrac{2}{3}nh^3\mathfrak{E}' + \frac{4nh^3}{3a}\mathfrak{E} + \frac{2\rho h^3}{3a}\mathrm{E}\,(a\mathrm{E}\ddot{\mathrm{K}} - \ddot{w} + \mathrm{Z}) \\
\mathrm{M}_2 &= 2nh\varpi + \tfrac{1}{3}nh^3p' + \frac{2nh^3}{3a}p \\
\mathrm{N}_2 &= \tfrac{4}{3}nh^3\left(\frac{d\mathfrak{E}}{dz} + \frac{1}{2a}\frac{dp}{d\phi}\right) + \frac{4nh^3}{3a}\frac{d\mathfrak{A}}{dz} + \tfrac{2}{3}\rho h^3\left(\frac{d\ddot{w}}{dz} - \frac{\ddot{u}}{a} + \frac{\mathrm{X}}{a}\right) \\
\mathrm{G}_2 &= \tfrac{4}{3}nh^3\left(\mathfrak{E} + \frac{\mathfrak{A}}{a}\right) \\
\mathrm{H}_1 &= -\tfrac{2}{3}nh^3\left(p + \frac{\varpi}{a}\right)
\end{aligned}
\right\} \qquad (43)
$$

which give the values of the sectional stresses across a circular section ; and

$$
\left.
\begin{aligned}
\mathrm{M}_1 &= 2nh\varpi + \tfrac{1}{3}nh^3p' \\
\mathrm{T}_2 &= 4nh\mathfrak{B} - \frac{4nh^3}{3a}\mathrm{E}\mathfrak{F} + \tfrac{2}{3}nh^3\mathfrak{F}' + \frac{2\rho h^3}{3a}\mathrm{E}\,(\alpha\mathrm{E}\ddot{\mathrm{K}} - \ddot{w} + \mathrm{Z}) \\
\mathrm{N}_1 &= \tfrac{4}{3}nh^3\left(\frac{1}{a}\frac{d\mathfrak{F}}{d\phi} + \tfrac{1}{2}\frac{dp}{dz}\right) + \frac{2nh^3}{3a}\frac{d\varpi}{dz} + \frac{2\rho h^3}{3a}\left(\frac{d\ddot{w}}{d\phi} - 2\ddot{v} + \mathrm{Y}\right) \\
\mathrm{G}_1 &= -\tfrac{4}{3}nh^3\mathfrak{F} \\
\mathrm{H}_2 &= \tfrac{2}{3}nh^3p
\end{aligned}
\right\} \quad . \quad . \quad . \quad (44)
$$

which give the values of the sectional stresses across a meridian.

If we compare these equations with the third and fourth of (12), with (14), and with the fourth and fifth of (11), we see that we have reproduced (i.) the values of M_1, M_2 given by (12); (ii.) the values of the couples given by (14); (iii.) the values of the normal shearing stresses which are obtained from the fourth and fifth of (11), by substituting the values of the couples from (14). We have thus subjected our fundamental hypothesis to a fairly searching test. It is, however, in our power to subject it to a still further test; for if we equate the coefficients of δu, δv, δw in the surface integrals in (40) and (42), we shall obtain the equations of motion in terms of the displacements, and on substituting the values of the sectional stresses from (43) and (44), in the first three of (11), we ought to reproduce the equations of motion in terms of the displacements which we have obtained from the variational equation.

From (30), (36), (40), and (42) it follows that these equations are

$$\rho\left\{2(\ddot{u}-\mathrm{X})+\tfrac{1}{3}h^2\mathrm{E}\frac{d\ddot{\mathrm{K}}}{dz}-\frac{2h^2}{3a}\frac{d\ddot{w}}{dz}\right\}=4n\left(\frac{d\mathfrak{A}}{dz}+\frac{1}{2a}\frac{d\varpi}{d\phi}\right)-\frac{4nh^2}{3a}\mathrm{E}\frac{d\mathfrak{F}}{dz}$$

$$+\tfrac{2}{3}nh^2\left(\frac{d\mathfrak{E}'}{dz}+\frac{1}{2a}\frac{dp'}{d\phi}\right)+\frac{4nh^2}{3a}\frac{d\mathfrak{E}}{dz}+\tfrac{2}{3}\rho h^2\frac{d\gamma}{dz} \qquad (45).$$

$$\rho\left\{2(\ddot{v}-\mathrm{Y})+\frac{h^2}{3a}\mathrm{E}\frac{d\ddot{\mathrm{K}}}{d\phi}+\frac{2h^2}{3a}\left(\ddot{v}-\frac{d\ddot{w}}{d\phi}\right)\right\}$$

$$=4n\left(\frac{1}{a}\frac{d\mathfrak{B}}{d\phi}+\tfrac{1}{2}\frac{d\varpi}{dz}\right)+\frac{4nh^2}{3a^2}\left\{(1-\mathrm{E})\frac{d\mathfrak{F}}{d\phi}+a\frac{dp}{dz}+\tfrac{1}{2}\frac{d\varpi}{dz}\right\}$$

$$+\tfrac{2}{3}nh^2\left(\frac{1}{a}\frac{d\mathfrak{F}'}{d\phi}+\tfrac{1}{2}\frac{dp'}{dz}\right)+\frac{2\rho h^2}{3a}\frac{d\gamma}{d\phi}+\frac{2\rho h^2}{3a^2}\left(\frac{d\ddot{w}}{d\phi}-2\ddot{v}+\mathrm{Y}\right) \quad . \quad . \quad (46).$$

$$\rho\left\{2(\ddot{w}-\mathrm{Z})-\tfrac{1}{3}h^2\mathrm{E}(\ddot{\lambda}+\ddot{\mu})-\frac{2h^2}{3a}\mathrm{E}\ddot{\mathrm{K}}\right\}$$

$$=-\frac{4n\mathfrak{B}}{a}+\frac{4nh^2}{3a^2}\left\{a^2\frac{d^2\mathfrak{E}}{dz^2}+\frac{d^2\mathfrak{F}}{d\phi^2}+\mathrm{E}\mathfrak{F}+a\frac{d^2p}{dz\,d\phi}\right\}$$

$$-\frac{2nh^2}{3a}\mathfrak{F}'+\frac{4nh^2}{3a}\frac{d}{dz}\left(\frac{d\mathfrak{A}}{dz}+\frac{1}{2a}\frac{d\varpi}{d\phi}\right)+\tfrac{2}{3}\rho h^2\frac{d}{dz}\left(\frac{d\ddot{w}}{dz}-\frac{\ddot{u}}{a}+\frac{\mathrm{X}}{a}\right)$$

$$+\frac{2\rho h^2}{3a^2}\frac{d}{d\phi}\left(\frac{d\ddot{w}}{d\phi}-2\ddot{v}+\mathrm{Y}\right)-\frac{2\rho h^2}{3a^2}\mathrm{E}(a\mathrm{E}\ddot{\mathrm{K}}-\ddot{w}+\mathrm{Z}) \quad . \quad . \quad . \quad . \quad (47).$$

If we compare these equations with the equations obtained by substituting the values of the sectional stresses in the first line of (11), it will be found that they agree in every respect.

10. It will hereafter be necessary to consider certain problems in which the middle surface is supposed to experience no extension or contraction throughout the motion; and it will, therefore, be necessary to obtain the requisite equations when this is supposed to be the case.

The conditions of inextensibility are

$$\sigma_1=0, \quad \sigma_2=0, \quad \varpi=0;$$

or

$$\frac{du}{dz}=0, \qquad \frac{dv}{d\phi}+w=0, \qquad \frac{du}{d\phi}+a\frac{dv}{dz}=0 \quad . \quad . \quad . \quad . \quad (48),$$

which require that

$$u = \mathrm{U}a, \qquad v = -z\frac{d\mathrm{U}}{d\phi} + \mathrm{V}, \qquad w = z\frac{d^2\mathrm{U}}{d\phi^2} - \frac{d\mathrm{V}}{d\phi} \quad . \quad . \quad . \quad . \quad (49),$$

where U and V are functions of ϕ alone.

In this case the potential energy reduces to the second line alone, and from (15), (16), and (17) we obtain

$$\left.\begin{aligned} \lambda &= 0 \\ \mu &= -\frac{1}{a^2}\left(\frac{d^2w}{d\phi^2} + w\right) \\ p &= -\frac{2}{a}\left(\frac{d^2w}{dzd\phi} - \frac{dv}{dz}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

and from (24)

$$\mathrm{W} = \frac{4nh^3}{3a^2}\left\{\frac{m}{a^2(m+n)}\left(\frac{d^2w}{d\phi^2} + w\right)^2 + \left(\frac{d^2w}{dzd\phi} - \frac{dv}{dz}\right)^2\right\} \quad . \quad . \quad . \quad . \quad (51),$$

which agrees with the expression obtained by Lord RAYLEIGH.[*]

Also from (14)

$$\left.\begin{aligned} \mathrm{G}_1 &= \frac{4nh^3}{3a^2}(1+\mathrm{E})\left(\frac{d^2w}{d\phi^2} + w\right) \\ \mathrm{G}_2 &= -\frac{4nh^3}{3a^2}\mathrm{E}\left(\frac{d^2w}{d\phi^2} + w\right) \\ \mathrm{H}_1 &= -\mathrm{H}_2 = -\tfrac{2}{3}nh^3p \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52).$$

The values of the stresses M_1, M_2 may be obtained either from (43) and (44), or from (12) combined with (15), (16), (17), and (18) by introducing the conditions of inextensibility; and the values of T_1, T_2 might be calculated by taking the variation subject to the conditions of inextensibility, and using indeterminate multipliers. This process would not, however, be of much assistance, inasmuch as it would introduce two undetermined quantities into the values of T_1, T_2, which depend upon the boundary conditions; whereas in this case the values of T_1, T_2 can be obtained directly from the first and third of (11) combined with (49). The values of N_1, N_2 are given by the fourth and fifth of (11) combined with (50) and (52).

11. We must lastly consider the boundary conditions.

Equations (43) and (44) determine the stresses on the line elements $ad\phi$ and dz respectively, which are produced by the action of contiguous portions of the shell; and it might at first sight appear, as was supposed by POISSON,[†] that when a shell of finite dimensions is under the influence of forces and couples applied to its edges, these equations would give the values of such forces or couples, and that the conditions to be satisfied at a free edge would require that each of the above five stresses should vanish at a free edge. Kirchhoff[*] has, however, shown that this is not the case, but that the boundary conditions are only *four* in number; and the reason of this is, that it is possible to apply a certain distribution of stress to the edge of a shell, without producing any alteration in the potential energy.

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[*] 'Roy. Soc. Proc.,' vol. 45, p. 116.

[†] 'Paris, Acad. des Sciences, Mémoires,' 1829, vol. 8, p. 357.

By Stokes' theorem,

$$\int\left(\frac{dH'}{d\phi}\,\delta w + H'\,\frac{d\delta w}{d\phi}\right)d\phi + \int\left(\frac{dH'}{dz}\,\delta w + H'\,\frac{d\delta w}{dz}\right)dz = 0\,;$$

the integration extending round any curvilinear rectangle bounded by four lines of curvature OA, AD, DB, BA. If, therefore, we apply to the side AD the stresses

$$M_2' = H'/a, \qquad N_2' = \frac{1}{a}\,\frac{dH'}{d\phi}, \qquad H_1' = H'\,;$$

to the side DB the stresses

$$N_1' = \frac{dH'}{dz}, \qquad H_2' = -H',$$

and to the sides BO, OA, corresponding and opposite stresses respectively, the preceding integral becomes

$$\int\left\{M_2'\,\delta v + N_2'\,\delta w + \frac{H_1'}{a}\left(\frac{d\delta w}{d\phi} - \delta v\right)\right\}a\,d\phi + \int\left(N_1'\,\delta w - H_2'\,\frac{d\delta w}{dz}\right)dz = 0,$$

which shows that the work done by these stresses is zero. Such a system of stresses may, therefore, be applied or removed without interfering with the equilibrium or motion of the shell.

Let us now suppose that the rectangle OADB, instead of being under the action of the remainder of the shell, is isolated, and that its state of strain is maintained by means of constraining stresses applied to its edges; then it follows that if, instead of the torsional couples H_1, H_2, due to the action of contiguous portions of the shell, we apply torsional couples $\mathfrak{H}_1$, $\mathfrak{H}_2$, where

$$\mathfrak{H}_1 = H_1 + H' \qquad \text{. . . . .} \quad (53),$$

$$\mathfrak{H}_2 = H_2 - H' \qquad \text{. . . . .} \quad (54),$$

---

[*] 'Crelle,' vol. 40, p. 51, 1850, and Collected Works, p. 237.

the state of strain will remain unchanged, provided we apply in addition the stresses

$$\left.\begin{aligned}\mathfrak{M}_2 &= M_2 + H'/a \\ \mathfrak{N}_2 &= N_2 + \frac{1}{a}\frac{dH'}{d\phi}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

and

$$\mathfrak{N}_1 = N_1 + \frac{dH'}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56),$$

whence, eliminating H′ between (53), (55), and (54), (56) respectively, we obtain

$$\left.\begin{aligned}\mathfrak{M}_2 a - \mathfrak{H}_1 &= M_2 a - H_1 \\ \mathfrak{N}_2 - \frac{1}{a}\frac{d\mathfrak{H}_1}{d\phi} &= N_2 - \frac{1}{a}\frac{dH_1}{d\phi}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57),$$

and

$$\mathfrak{N}_1 + \frac{d\mathfrak{H}_2}{dz} = N_1 + \frac{dH_2}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58).$$

In these equations the Roman letters denote the stresses due to the action of contiguous portions of the shell, whose values are given by (43) and (44), whilst the Old English letters denote the values of the actual stresses applied to the boundary. If, therefore, the shell consists of a portion of a cylinder which is bounded by four lines of curvature and whose edges are free, the boundary conditions along the circular edges are obtained by equating the right hand sides of the first and fourth of (43), and the right hand sides of (57) to zero, the first two of which express the condition that the tension perpendicular to, and the flexural couple about, a line element of the circular edge must vanish when the edge is free; and the boundary conditions along the straight edge are similarly obtained by equating the right hand sides of the first, second, and fourth of (44), and the right hand side of (58) to zero, the first three of which express the conditions that the tangential shearing stress, the tension and the flexural couple must vanish when the free edge is a generating line. We may also, if we do not wish to introduce the time and the bodily forces into these equations, substitute for $\ddot{u} - X$, $\ddot{v} - Y$, $\ddot{w} - Z$ their approximate values from (32).

12. We have now obtained all the materials we require, for a perfectly accurate approximate solution of any problem relating to the vibrations of a thin cylindrical shell as far as the terms involving the cube of the thickness, but before proceeding to discuss any problems, it will be necessary to make some remarks respecting Mr. LOVE's paper. The first line of my expression for the potential energy which is given in (24), and which involves h and not h^3, agrees with the expression obtained by Mr. LOVE and other writers; also the approximate equations of motion (32) agree, as has been already pointed out, with the corresponding equations obtained by him, and by means of which he has discussed the extensional vibrations of a cylinder. It will

also hereafter appear, that observations of a precisely similar character apply to the corresponding equations which determine to a first approximation the extensional vibrations of a spherical shell. This portion of his paper therefore appears to be perfectly satisfactory; but that portion which involves the terms depending upon the cube of the thickness is open to criticism.

In the first place, he appears to have employed a system of *rectangular* axes, consisting of the normal at a point on the middle surface, and the tangents to the two lines of curvature through that point. Now, although it is immaterial, so long as we confine our attention to infinitesimals of the first order, whether a quantity is measured along the tangents to three orthogonal curves or along the curves themselves, yet when it is necessary to take into consideration infinitesimals of higher orders, which is always the case whenever an investigation involves changes of curvature, a method in which everything is referred to rectangular axes requires care; and on comparing the terms in h^3 in (24) with the corresponding terms in Mr. LOVE's expression for the potential energy, it will be seen that he has omitted several terms which involve the extensions of the middle surface, which partly, although not entirely, arises from his having omitted the factor $1 + h'/a$. It is not improbable that these terms may be small, but at the same time we are not at liberty to neglect them altogether; for it is quite evident that a term such as $\delta(\mathfrak{B}\mu)$ in the variational equation, will give rise to terms in the equations of motion and the equations giving the values of the sectional stresses, which do not involve the extension of the middle surface.

In the second place, on comparing Mr. LOVE's variational equation of motion* with my equations (25), (26), (28), and (29), it will be seen that he has omitted several terms in the expressions for $\delta\mathfrak{T}$, $\delta\mathrm{U}$, and $\delta\mathfrak{L}$.

In the third place he states (p. 521) that the extensional quantities "σ_1, σ_2, ϖ may not, in general, be regarded as of a higher order of small quantities than κ_2, λ_1, κ_1," which are the quantities upon which the bending depends. The argument of Lord RAYLEIGH† appears to me to show, that at points whose distance from the edge is large in comparison with the thickness, the extensional terms are usually small in comparison with the terms upon which the bending depends. It must be obvious to every one, that a thin plate of metal or a steel spring can be bent with the greatest ease by means of the fingers, whereas the production of any extension of the middle surface which would be capable of measurement, would involve considerable muscular effort. These considerations indicate that when a thin shell is vibrating, the change of curvature is so greatly in excess of the extension of the middle surface, that notwithstanding the smallness of h^3 compared with h, the product $h^3(\delta\rho^{-1})^2$ is large‡ compared with the product $h\sigma^2$. At the same time, inasmuch as the production of change of curvature involves some extension or contraction of all but the central layers, and consequently of those portions of the shell which are near its external surface, it does not seem unreasonable to suppose that in the neighbourhood of a free edge, an extension or contraction of the middle surface may take place, which is comparable with the change of curvature.

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* 'Phil. Trans.,' A., 1888, p. 514, equation (19).

† 'Roy. Soc. Proc.,' vol. 45, p. 105.

‡ The problem discussed in § 14 shows that the product $h\sigma^2$ *may be* of the order $h^5(\delta\rho^{-1})^2$, except in the neighbourhood of a free edge; but in the equations of motion we have to deal with the quantities $h\sigma$ and $h^3\,\delta\rho^{-1}$.

In the fourth place, Mr. LOVE appears to have argued as if the equations of motion of a shell, whose middle surface undergoes no extension or contraction throughout the motion, might be obtained from his general equations (30), (31), (32), by putting $\sigma_1 = \sigma_2 = \varpi = 0$; but it has already been pointed out, that the correct equations for this kind of motion must be obtained by taking the variation subject to the conditions of inextensibility, and introducing indeterminate multipliers. It will be shown in the next section, that in the case of the flexural vibrations of an indefinitely long complete cylindrical shell considered by HOPPE and Lord RAYLEIGH,[*] the differential equation for the tangential displacement v is of the sixth degree, and that when the cross section of the shell consists of a circular arc, this equation contains sufficient constants to enable all the conditions of the problem to be satisfied.

13. The first problem which we shall consider will be that of the flexural vibrations of an indefinitely long cylinder, in which the displacement of every element lies in a plane perpendicular to the axis of the cylinder, and which has been discussed by HOPPE and Lord RAYLEIGH.

In this problem the middle surface is supposed to undergo no extension or contraction throughout the motion, and the solution is most easily obtained by means of the general equations (11). In these equations we must omit all the terms on the right hand sides which involve h^3, for they would, if retained, give rise to a term involving h^4 in the period equation, which must be rejected, since we do not carry the approximation further than h^2 in determining the period.

We evidently have[†] $M_1 = N_2 = H_2 = 0$; also none of the quantities are functions of z. The equations of motion are thus

$$\frac{dT_2}{d\phi} + N_1 = 2\rho ha\ddot{v},$$

$$\frac{dN_1}{d\phi} - T_2 = 2\rho ha\ddot{w},$$

$$\frac{dG_1}{d\phi} + N_1 a = 0,$$

---

[*] 'Theory of Sound,' vol. 1, p. 324; 'Roy. Soc. Proc.,' vol. 45, p. 129. Equation (51).

[†] We shall presently see that these conditions imply a constraint at infinity.

also the condition of inextensibility gives

$$\frac{dv}{d\phi} + w = 0.$$

Eliminating N_1, T_2, and w, and substituting the value of G_1 from the first of (52), we obtain

$$\frac{4mnh^2}{3\rho a^4 (m+n)} \left(\frac{d^3}{d\phi^3} + \frac{d}{d\phi}\right)^2 v + \frac{d^2 \ddot{v}}{d\phi^2} - \ddot{v} = 0 \quad \text{. . . . . . (59),}$$

whence, putting

$$v = \mathrm{A}\epsilon^{\iota pt + \iota s\phi},$$

we obtain

$$p^2 = \frac{4mnh^2 s^2 (s^2 - 1)^2}{3\rho a^4 (m+n)(s^2+1)} \quad \text{. . . . . (60),}$$

which is the required result.

If the cylinder is complete, s is any integer, unity excluded, but if the cross-section of the cylinder consists of a circular arc of length $2a\alpha$, s will not be an integer. Its values in terms of p are the six roots of (60), but in order to obtain the frequency equation, the value of s in terms of the dimensions and elastic constants is required. The additional equations are obtained from the boundary conditions, which have to be satisfied along the straight edges of the shell, and these require that the tension T_2, the normal shearing stress N_1, and the flexural couple G_1 should vanish at the edges where $\phi = \pm\,\alpha$.

Since

$$G_1 = -\frac{8mnh^3}{3(m+n)}\mu,$$

where

$$\mu = \frac{1}{a^2}\left(\frac{d^3 v}{d\phi^3} + \frac{dv}{d\phi}\right),$$

the boundary conditions are obviously

$$\mu = 0,$$

$$\frac{d\mu}{d\phi} = 0,$$

$$\frac{4mnh^2}{3\rho a^2 (m+n)}\,\frac{d^2\mu}{d\phi^2} + \frac{d^3 v}{d\phi\, dt^2} = 0.$$

These conditions have to be satisfied at each of the edges of the shell where $\phi = \pm\,\alpha$, and there are, therefore, six equations of condition; hence the six constants which appear in the solution of (59) can be eliminated, and the resulting determinantal equation, combined with (60), will give the frequency.*

If a complete cylinder of *finite length* were vibrating in this manner, it would be necessary to satisfy the conditions at the circular ends, and this would require that $T_1 = 0$, $G_2 = 0$ at the ends for all values of ϕ; and from the first and fourth of (43) we see that this requires that $\mu = 0$, or

$$\frac{d^2w}{d\phi^2} + w = 0,$$

whence

$$w = A\cos\phi + B\sin\phi$$

for all values of ϕ. Since it is impossible to satisfy this condition for the kind of motion considered, it follows that when the cylinder is of finite length it would be necessary to apply at every point of the circular boundary a tension T_1 and a couple G_2 of the requisite amount.

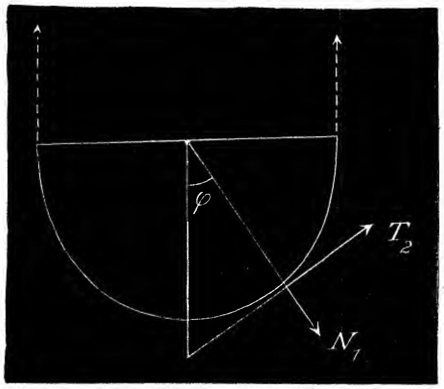
This is the question upon which Lord RAYLEIGH and Mr. LOVE are at issue; and the preceding investigation shows that Mr. LOVE is right in supposing that it is impossible to satisfy the boundary conditions along the curved edges of a cylindrical shell when these edges are free, although he does not appear to have noticed that it is possible to satisfy these conditions when the free edges are generating lines. In order to obtain a complete mathematical solution of this question, it would be necessary to work out the problem of the free vibrations of a complete cylindrical shell of given length $2l$, which is deformed in such a manner that $dv/d\phi + w = 0$, where v and w are functions of ϕ alone, and is then let go, without assuming that the middle surface remains unextended during the subsequent motion.

Owing unfortunately to the extremely complicated nature of the general equations, a rigorous solution of this problem would be exceedingly difficult. We shall, however, be able to throw some light upon this question, by solving and discussing the following much simpler statical problem.

14. Let us consider a heavy cylindrical shell, whose cross section is a semicircle, and which is suspended by means of vertical bands attached to its straight edges, so that its axis is horizontal; and let us investigate the state of strain produced by its own weight.

In order to simplify the problem as much as possible, we shall suppose that the displacement of every point of the middle surface lies in a plane perpendicular to the axis, and we shall afterwards investigate the stresses which must be applied to the circular edges, in order to maintain this state of things.

* [This problem is of a similar character to that of a bar, whose natural form is circular, and which has been discussed by LAMB. 'London Math. Soc. Proc.,' vol. 19, p. 365.—June, 1890.]



We have

$$Y = -g \sin \phi, \quad Z = g \cos \phi ;$$

whence, if $W = 2g\rho a h$, the equations of equilibrium are

$$\frac{dT_2}{d\phi} + N_1 = W \sin \phi ,$$

$$\frac{dN_1}{d\phi} - T_2 = - W \cos \phi ,$$

$$\frac{dG_1}{d\phi} + N_1 a = - \frac{h^2}{3a} W \sin \phi ;$$

from which we obtain

$$\frac{d^2T_2}{d\phi^2} + T_2 = 2W \cos \phi ,$$

the integral of which is

$$T_2 = A \cos \phi + B \sin \phi + W\phi \sin \phi ,$$

and, therefore,

$$N_1 = A \sin \phi - B \cos \phi - W\phi \cos \phi.$$

Since $N_1 = 0$ when $\phi = \frac{1}{2}\pi$, $A = 0$; also since $T_2 = \frac{1}{2}\pi W$ when $\phi = \frac{1}{2}\pi$, $B = 0$; whence

$$T_2 = W\phi \sin \phi, \qquad N_1 = - W\phi \cos \phi \quad . \; . \; . \; . \; . \; . \; . \quad (61),$$

and, therefore,

$$\frac{dG_1}{d\phi} = Wa \left(\phi \cos \phi - \frac{h^2}{3a^2} \sin \phi\right),$$

whence

$$G_1 = Wa \left(\phi \sin \phi + \cos \phi + \frac{h^2}{3a^2} \cos \phi\right) + C.$$

Since $G_1 = 0$ when $\phi = \frac{1}{2}\pi$, $C = -\frac{1}{2}W\pi a$; accordingly

$$G_1 = Wa \left\{\phi \sin \phi + \left(1 + \frac{h^2}{3a^2}\right) \cos \phi - \tfrac{1}{2}\pi\right\} \quad . \; . \; . \; . \; . \quad (62).$$

But

$$G_1 = -\tfrac{4}{3}nh^3\mathfrak{F} = -\tfrac{4}{3}nh^3(1+E)\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63),$$

whence

$$\tfrac{4}{3}nh^3\mu = -\frac{Wa}{1+E}\left\{\phi\sin\phi + \left(1+\frac{h^2}{3a^2}\right)\cos\phi - \tfrac{1}{2}\pi\right\}. \quad . \quad . \quad . \quad (64).$$

Again, if R denote the change of curvature along a circular section, so that

$$R = -\frac{1}{a^2}\left(\frac{d^2w}{d\phi^2} + w\right)$$

we have

$$\mu = R - E\sigma_2/a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65).$$

Also by (18)

$$\mu' = -(2+E)\mu/a + \frac{E}{a^2}\frac{d^2\sigma_2}{d\phi^2}$$

and, therefore,

$$\mathfrak{F}' = (1+E)\mu' = -(1+E)(2+E)\mu/a + \frac{(1+E)E}{a^2}\frac{d^2\sigma_2}{d\phi^2},$$

whence, by the second of (44),

$$T_2 = 4nh(1+E)\sigma_2 - \frac{2nh^3}{3a}(1+E)(3+2E)\mu + \frac{2nh^3}{3a^2}(1+E)E\frac{d^2\sigma_2}{d\phi^2} + \frac{h^2}{3a^2}EW\cos\phi.$$

Substituting the values of T_2 and μ from (61) and (64), we obtain

$$2nh(1+E)\left\{2\sigma_2 + \frac{h^2E}{3a^2}\frac{d^2\sigma_2}{d\phi^2}\right\} + \tfrac{1}{2}(2+3E)\left\{\phi\sin\phi + \left(1+\frac{h^2}{3a^2}\right)\cos\phi - \tfrac{1}{2}\pi\right\}$$
$$+ \frac{h^2E}{3a^2}W\cos\phi - W\phi\sin\phi = 0.$$

This equation might, if necessary, be solved by successive approximation, but a first approximation will be sufficient. Omitting the terms in h^2, and recollecting that W involves h as a factor, we obtain

$$4nh(1+E)\sigma_2 + W(\cos\phi - \tfrac{1}{2}\pi) + \tfrac{3}{2}EW(\phi\sin\phi + \cos\phi - \tfrac{1}{2}\pi) = 0 \quad . \quad (66),$$

whence from (64), (65), and (66), we obtain

$$\frac{R}{\sigma_2} = \frac{E}{a} + \frac{3a(\tfrac{1}{2}\pi - \phi\sin\phi - \cos\phi)}{h^2\{\tfrac{1}{2}\pi - \cos\phi + \tfrac{3}{2}E(\tfrac{1}{2}\pi - \phi\sin\phi - \cos\phi)\}} \quad . \quad . \quad . \quad . \quad (67).$$

Since the numerator of this fraction is an even function of ϕ, it does not change sign with ϕ; also the numerator is always positive between the limits $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$, and its maximum value occurs when $\phi = 0$ and is equal to $\frac{1}{2}\pi - 1$, and its minimum value occurs when $\phi = \frac{1}{2}\pi$ and is equal to zero. We, therefore, see that when $\phi = 0$,

$$\frac{\text{R}}{\sigma_2} = \frac{\text{E}}{a} + \frac{3a}{h^2};$$

and when $\phi = \frac{1}{2}\pi$,

$$\frac{\text{R}}{\sigma_2} = \frac{\text{E}}{a}.$$

Since the thickness of the shell is supposed to be small compared with its radius, it follows that the change of curvature is large compared with the extension of the middle surface, except when $a(\frac{1}{2}\pi - \phi)$ is comparable with h, *i.e.*, in the neighbourhood of the straight edges of the shell; and therefore at all points of the shell whose distances from the edges are large in comparison with its thickness, the terms depending upon the product of the change of curvature and the cube of the thickness, *i.e.*, the terms upon which the bending depends, are of the same order as the terms depending upon the product of the extension of the middle surface and the thickness; but at points whose distances from the edge are comparable with the thickness of the shell, the extension of the middle surface is of the same order as the change of curvature, and therefore the terms depending upon the product of the change of curvature and the cube of the thickness are small in comparison with the terms depending upon the product of the extension and the thickness.

We shall now calculate the stresses which must be applied to the circular edges in order to maintain this particular kind of strain. From (43) we have

$$\text{G}_2 = \tfrac{4}{3} nh^3 \text{E} (\mu + \sigma_2/a).$$

Substituting the values of μ and σ_2 from (62), (63), and (64), we see that the terms in σ_2 may be omitted, and we obtain

$$\text{G}_2 = \frac{\text{EW}a}{1 + \text{E}} (\tfrac{1}{2}\pi - \phi \sin\phi - \cos\phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68),$$

which shows that G_2 is positive.

Also

$$\begin{aligned} \text{T}_1 &= 4nh\,\text{E}\,\sigma_2 - \frac{2nh^3}{3a}\text{E}(2 + 3\text{E})\,\mu \\ &= \frac{\text{EW}}{1 + \text{E}}\phi \sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69), \end{aligned}$$

which shows that T_1 is positive.

Comparing (68) and (69) with (61) and (62) we see that ratios of the tension T_1 and the couple G_2, to T_2 and G_1 are numerically equal to $\text{E}/(1 + \text{E})$; we further see that G_1 is negative, and, therefore, the strain tends to increase the curvature of the circular sections. Now when a cylindrical shell is bent about a generating line in such a manner that its curvature is increased, all lines parallel to the axis which lie on the convex side of the middle surface will be contracted, whilst all such lines which lie on the concave side will be extended, and this contraction and extension will give rise to a couple about the circular sections which tends to produce anticlastic curvature of the generating lines. In order to prevent this taking place it is necessary to apply at every point of the circular edges a couple G_2 tending to produce synclastic curvature, and a tension T_1 parallel to the axis, whose values are given by (68) and (69). If this couple and tension were removed, the middle surface would bend about its circular sections, and anticlastic curvature of the generating lines would be produced, and this would necessarily involve extension or contraction parallel to the axis, so that the problem could no longer be treated as one of two dimensions.

It must, however, be within the experience of everyone that when a thin cylindrical shell of finite length, whose cross section is the arc of a circle, is bent about its generating lines, the shell does not assume a saddle-back form, and consequently the anticlastic curvature of the generating lines must be so small as to be inappreciable. This circumstance furnishes an additional argument in favour of the supposition that the extension of the middle surface is only sensible in the neighbourhood of the free edges.

We therefore conclude that if the circular edges were free, some extension or contraction of the middle surface must necessarily take place, but that this extension or contraction is small compared with the change of curvature along a circular section, except just in the neighbourhood of the edges. From these considerations the inference is, that if by means of proper constraints applied to the circular edges, a cylindrical shell were enabled to execute the non-extensional vibrations discussed in § 13, the vibrations would cease to be non-extensional if the constraints were removed; but that the amplitudes of those portions of the displacements upon which the extension depends, would be very small compared with the amplitudes of those portions upon which the change of curvature along a circular section depends, except just in the neighbourhood of the edges. Moreover, the theory of plane plates shows, that the frequency of the extensional vibrations is expressible* by means of a series of even powers of h, *commencing with a term independent of* h; whilst the frequency of the flexural vibrations is expressible by means of a similar series *commencing with* h^2. It therefore follows, that the pitch of the notes arising from the former class of vibrations, is high compared with the pitch of those arising from the latter class. And although, except under special circumstances, it is not possible in the case of curved shells whose edges are free, for these two classes of vibrations to coexist independently, as in the case of a plane plate; yet recent investigations show, that the pitch of notes which mainly depend upon the extension is usually high in comparison with the pitch of notes which mainly depend upon bending, and consequently the notes arising from the former cause, both on account of the smallness of their amplitudes and the highness of their pitch, would probably be so feeble in comparison with those which arise from the latter cause, as to be scarcely capable of producing any appreciable effect upon the ear. Judging from the usual course of such investigations, the probable form of the exact solution of the problems suggested at the end of § 13 would be that of an infinite series, the periods of the different components of which would satisfy a transcendental equation having an infinite number of roots; but the preceding considerations point to the conclusion that the frequency of the gravest* note given by (60), viz., $p^2 = 48mnh^2/5\rho a^4 (m + n)$, although perhaps not rigorously accurate, is a close approximation to the truth.

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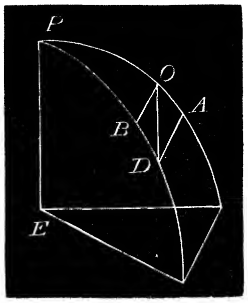
* Lord RAYLEIGH, 'London Math. Soc. Proc.,' vol. 20, p. 225. See especially equations (38), (45), and (53).

Spherical Shells.

15. The fundamental equations for a spherical shell can be investigated in precisely the same manner as in the case of a cylindrical shell.

If u', v', w' be the component displacements at any point of the substance of the shell in the directions, θ, ϕ, r, the equations connecting the displacements and strains are



$$\left.\begin{aligned}
\sigma'_1 &= \frac{1}{r}\left(\frac{du'}{d\theta} + w'\right) \\
\sigma'_2 &= \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{dv'}{d\phi} + u'\cot\theta + w'\right) \\
\sigma'_3 &= \frac{dw'}{dr} \\
\varpi'_1 &= \frac{1}{r\sin\theta}\frac{dw'}{d\phi} + \frac{dv'}{dr} - \frac{v'}{r} \\
\varpi'_2 &= \frac{du'}{dr} - \frac{u'}{r} + \frac{1}{r}\frac{dw'}{d\theta} \\
\varpi'_3 &= \frac{1}{r}\left(\frac{dv'}{d\theta} - v'\cot\theta + \frac{1}{\sin\theta}\frac{du'}{d\phi}\right)
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (1),$$

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* [The experiments of Lord RAYLEIGH, 'Phil. Mag.,' Jan., 1890, show that the effective pitch of a bell is usually not the same as that of its gravest tone; and, in the bells which he examined, the fifth tone in order was the one which agreed with the nominal pitch of the bell.—June, 1890.]

whence

$$\left.\begin{aligned}\left(\frac{du}{dr}\right) &= \varpi_2 + \frac{u}{a} - \frac{1}{a}\frac{dw}{d\theta}\\ \left(\frac{dv}{dr}\right) &= \varpi_1 + \frac{v}{a} - \frac{1}{a\sin\theta}\frac{dw}{d\phi}\\ \left(\frac{dw}{dr}\right) &= \frac{\mathrm{A}}{m+n} - \mathrm{EK}\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2),$$

also

$$\left.\begin{aligned}\left(\frac{d^2u}{dr^2}\right) &= \left(\frac{d\varpi_2}{dr}\right) + \frac{\varpi_2}{a} - \frac{1}{a(m+n)}\frac{d\mathrm{A}}{d\theta} + \frac{\mathrm{E}}{a}\frac{d\mathrm{K}}{d\theta}\\ \left(\frac{d^2v}{dr^2}\right) &= \left(\frac{d\varpi_1}{dr}\right) + \frac{\varpi_1}{a} - \frac{1}{a(m+n)\sin\phi}\frac{d\mathrm{A}}{d\phi} + \frac{\mathrm{E}}{a\sin\theta}\frac{d\mathrm{K}}{d\phi}\\ \left(\frac{d^2w}{dr^2}\right) &= \frac{\mathrm{A}_1}{m+n} - \mathrm{E}(\lambda+\mu)\end{aligned}\right\} \quad . \; . \; . \; . \quad (3).$$

16. We can now obtain the equations of motion in terms of the sectional stresses.

If $d\mathrm{S}$ be an element of the middle surface whose coordinates are (a, θ, ϕ), and $d\mathrm{S}'$ an element of a layer of the shell whose coordinates are $(a + h', \theta, \phi)$, then $d\mathrm{S}' = (1 + h'/a)^2\, d\mathrm{S}$; whence, if in the figure OA, OB respectively coincide with the meridians and circular sections, we obtain by resolving parallel to OA,

$$\frac{d}{d\theta}(\mathrm{T}_1 a\sin\theta\,\delta\phi)\,\delta\theta - \mathrm{T}_2 a\cos\theta\,\delta\theta\,\delta\phi + \frac{d}{d\phi}(\mathrm{M}_1 a\,\delta\theta)\,\delta\phi + \mathrm{N}_2 a\sin\theta\,\delta\theta\,\delta\phi$$

$$= \rho\, d\mathrm{S}\int_{-h}^{h}(\ddot{u}' - \mathrm{X})(1 + h'/a)^2\, dh' \quad . \; . \quad (4).$$

But

$$u' = u + h'\left(\frac{du}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2u}{dr^2}\right);$$

accordingly if we substitute the values of (du/dr) and (d^2u/dr^2) from (2) and (3), and recollect that all quantities which vanish with h may be omitted when multiplied by h^3, the right hand side of (4) becomes

$$\rho d\mathrm{S}\left\{2h\left(1 + \frac{h^2}{a^2}\right)\ddot{u} + \frac{h^3\mathrm{E}}{3a}\frac{d\ddot{\mathrm{K}}}{d\theta} - \frac{4h^3}{3a^2}\frac{d\ddot{w}}{d\theta} - 2h\left(1 + \frac{h^2}{3a^2}\right)\mathrm{X}\right\}.$$

Resolving parallel to OB, OC, and then taking moments about OA, OB, OC we shall obtain in a similar way five other equations, which, together with (4), may be written,

$$\left.\begin{aligned}
&\frac{d}{d\theta}(\mathrm{T}_1 \sin\theta) - \mathrm{T}_2\cos\theta + \frac{d\mathrm{M}_1}{d\phi} + \mathrm{N}_2\sin\theta \\
&\quad = \left\{2h\left(1+\frac{h^2}{a^2}\right)\ddot{u} + \frac{h^3\mathrm{E}}{3a}\frac{d\ddot{\mathrm{K}}}{d\theta} - \frac{4h^3}{3a^2}\frac{d\ddot{w}}{d\theta} - 2h\left(1+\frac{h^2}{3a^2}\right)\mathrm{X}\right\}\rho a\sin\theta, \\
&\frac{d\mathrm{T}_2}{d\phi} + \frac{d}{d\theta}(\mathrm{M}_2\sin\theta) + \mathrm{M}_1\cos\theta + \mathrm{N}_1\sin\theta \\
&\quad = \left\{2h\left(1+\frac{h^2}{a^2}\right)\ddot{v} + \frac{h^3\mathrm{E}}{3a\sin\theta}\frac{d\ddot{\mathrm{K}}}{d\phi} - \frac{4h^3}{3a^2\sin\theta}\frac{d\ddot{w}}{d\phi} - 2h\left(1+\frac{h^2}{3a^2}\right)\mathrm{Y}\right\}\rho a\sin\theta, \\
&\frac{d}{d\theta}(\mathrm{N}_2\sin\theta) + \frac{d\mathrm{N}_1}{d\phi} - (\mathrm{T}_1+\mathrm{T}_2)\sin\theta \\
&\quad = \left\{2h\left(1+\frac{h^2}{3a^2}\right)\ddot{w} - \tfrac{1}{3}h^3\mathrm{E}(\ddot{\lambda}+\ddot{\mu}) - \frac{4h^3}{3a}\mathrm{E}\ddot{\mathrm{K}} - 2h\left(1+\frac{h^2}{3a^2}\right)\mathrm{Z}\right\}\rho a\sin\theta, \\
&\frac{d\mathrm{G}_1}{d\phi} + \mathrm{N}_1 a\sin\theta + \frac{d}{d\theta}(\mathrm{H}_1\sin\theta) - \mathrm{H}_2\cos\theta \\
&\quad = \tfrac{2}{3}\rho h^3\left(\frac{1}{\sin\theta}\frac{d\ddot{w}}{d\theta} - 3\ddot{v} + 2\mathrm{Y}\right)\sin\theta, \\
&\frac{d}{d\theta}(\mathrm{G}_2\sin\theta) + \mathrm{G}_1\cos\theta - \mathrm{N}_2 a\sin\theta + \frac{d\mathrm{H}_2}{d\theta} \\
&\quad = -\tfrac{2}{3}\rho h^3\left(\frac{d\ddot{w}}{d\theta} - 3\ddot{u} + 2\mathrm{X}\right)\sin\theta, \\
&\qquad\qquad (\mathrm{M}_1 - \mathrm{M}_2)a - \mathrm{H}_1 - \mathrm{H}_2 = 0.
\end{aligned}\right\} \quad (5)$$

17. We shall now (as in the case of a cylindrical shell) proceed to obtain the values of the couples and the stresses M_1, M_2 by direct calculation.

We have

$$\mathrm{T}_1\, a\sin\theta\,\delta\phi = \int_{-h}^{h}\mathrm{P}'(a+h')\sin\theta\,\delta\phi\, dh';$$

whence

$$\left.\begin{aligned}
\mathrm{T}_1 &= 2h\mathrm{P} + \tfrac{1}{3}h^3\left(\frac{d^2\mathrm{P}}{dr^2}\right) + \frac{2h^3}{3a}\left(\frac{d\mathrm{P}}{dr}\right) \\
\mathrm{T}_2 &= 2h\mathrm{Q} + \tfrac{1}{3}h^3\left(\frac{d^2\mathrm{Q}}{dr^2}\right) + \frac{2h^3}{3a}\left(\frac{d\mathrm{Q}}{dr}\right) \\
\mathrm{M}_1 = \mathrm{M}_2 &= 2nh\varpi + \tfrac{1}{3}nh^3\left(\frac{d^2\varpi_3}{dr^2}\right) + \frac{2nh_3}{3a}\left(\frac{d\varpi_3}{dr}\right) \\
\mathrm{G}_1 &= -\tfrac{2}{3}h^3\left\{\left(\frac{d\mathrm{Q}}{dr}\right) + \frac{\mathrm{Q}}{a}\right] \\
\mathrm{G}_2 &= \tfrac{2}{3}h^3\left\{\left(\frac{d\mathrm{P}}{dr}\right) + \frac{\mathrm{P}}{a}\right] \\
\mathrm{H}_1 = -\mathrm{H}_2 &= -\tfrac{2}{3}nh^3\left\{\left(\frac{d\varpi_3}{dr}\right) + \frac{\varpi_3}{a}\right\}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6).$$

The third and sixth of these equations satisfy the last of (5) as ought to be the case. Also, employing our previous notation, we see that

$$\left.\begin{aligned} G_1 = -\tfrac{4}{3}nh^3\left(\mathfrak{F} + \frac{\mathfrak{B}}{a}\right),\quad G_2 = \tfrac{4}{3}nh^3\left(\mathfrak{E} + \frac{\mathfrak{A}}{a}\right) \\ H_1 = -H_2 = -\tfrac{2}{3}nh^3\left(p + \frac{\varpi_3}{a}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Since the couples are proportional to the cube of the thickness, it follows from the fourth and fifth of (5) that the normal shearing stresses N_1, N_2 are also proportional to the cube of the thickness, and, therefore, that the shearing strains ϖ'_1, ϖ'_2 are quadratic functions of h and h'.

Employing our previous notation, the next thing is to calculate the quantities λ, μ, p, λ', μ', p'. We have

$$\left.\begin{aligned} \lambda &= \left(\frac{d\sigma_1}{dr}\right) = -\frac{1}{a^2}\left(\frac{d^2w}{d\theta^2} + w\right) - \frac{E}{a}(\sigma_1 + \sigma_2) \\ \mu &= \left(\frac{d\sigma_2}{dr}\right) = -\frac{1}{a^2}\left(\frac{1}{\sin^2\theta}\frac{d^2w}{d\phi^2} + \cot\theta\frac{dw}{d\theta} + w\right) - \frac{E}{a}(\sigma_1 + \sigma_2) \\ p &= \left(\frac{d\varpi_3}{dr}\right) = \frac{2}{a^2\sin\theta}\left(\cot\theta\frac{dw}{d\phi} - \frac{d^2w}{d\theta\, d\phi}\right) \end{aligned}\right\} \quad . \quad . \quad (8),$$

in which equations we have omitted all quantities which vanish with h, because λ, μ, p occur in expressions which are multiplied by h^3. Similarly

$$\left.\begin{aligned} \lambda' &= -\frac{2\lambda}{a} - \frac{E}{a}(\lambda + \mu) + \frac{E}{a^2}\frac{d^2K}{d\theta^2} \\ \mu' &= -\frac{2\mu}{a} - \frac{E}{a}(\lambda + \mu) + \frac{E}{a^2}\left(\frac{1}{\sin^2\theta}\frac{d^2K}{d\phi^2} + \cot\theta\frac{dK}{d\theta}\right) \\ p' &= -\frac{2p}{a} - \frac{2E}{a^2\sin\theta}\left(\cot\theta\frac{dK}{d\phi} - \frac{d^2K}{d\theta\, d\phi}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad (9).$$

18. The variational equation may be written

$$\delta W + \delta\mathfrak{T} = \delta U + \delta\mathfrak{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

and we must now calculate the values of the four terms in it, and we shall begin with W.

Since we may omit ϖ'_1, ϖ'_2, and may, therefore, write ϖ for ϖ_3, the potential energy of any portion of the shell is

$$W = \tfrac{1}{2}\iiint_{-h}^{h}\left[(m+n)\Delta'^2 + n\left\{\varpi'^2 - 4(\sigma'_1\sigma'_2 + \sigma'_2\sigma'_3 + \sigma'_3\sigma'_1)\right\}\right](1 + h'/a)^2\, dh'\, dS \quad (11)$$

where the integration with respect to S extends over the middle surface of the portion considered. Since

$$\Delta' = \Delta + h'\left(\frac{d\Delta}{dr}\right) + \tfrac{1}{2}h'^2\left(\frac{d^2\Delta}{dr^2}\right) + \quad . \quad . \quad .$$

we obtain

$$\begin{aligned}
&\tfrac{1}{2}(m+n)\int_{-h}^{h}\Delta'^2\,(1+h'/a)^2\,dh' \\
&\quad = (m+n)\left\{h\left(1+\frac{h^2}{3a^2}\right)\Delta^2 + \tfrac{1}{3}h^3\left(\frac{d\Delta}{dr}\right)^2 + \tfrac{1}{3}h^3\Delta\left(\frac{d^2\Delta}{dr^2}\right) + \frac{4h^3}{3a}\Delta\left(\frac{d\Delta}{dr}\right)\right\} \\
&\quad = \frac{4n^2}{m+n}\left\{h\left(1+\frac{h^2}{3a^2}\right)(\sigma_1+\sigma_2+\mathrm{A}/2n)^2 + \tfrac{1}{3}h^3(\lambda+\mu)^2\right. \\
&\qquad\quad \left. + \tfrac{1}{3}h^3(\sigma_1+\sigma_2)(\lambda'+\mu') + \frac{4h^3}{3a}(\sigma_1+\sigma_2)(\lambda+\mu)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12),
\end{aligned}$$

also

$$\begin{aligned}
2n\int_{-h}^{h}\sigma'_1\sigma'_2\,(1+h'/a)^2\,dh' &= 4nh\,(1+\tfrac{1}{3}h^2/a^2)\,\sigma_1\sigma_2 + \tfrac{4}{3}\,nh^3\lambda\mu \\
&\quad + \tfrac{2}{3}\,nh^3\,(\lambda'\sigma_2+\mu'\sigma_1) + \frac{8nh^3}{3a}(\lambda\sigma_2+\mu\sigma_1) \quad . \quad . \qquad (13),
\end{aligned}$$

and

$$\begin{aligned}
&2n\int_{-h}^{h}(\sigma'_1+\sigma'_2)\,\sigma'_3\,(1+h'/a)^2\,dh' \\
&\quad = 4nh\,(1+\tfrac{1}{3}h^2/a^2)\left\{\frac{\mathrm{A}}{m+n} - \mathrm{E}\,(\sigma_1+\sigma_2)\right\}(\sigma_1+\sigma_2) - \tfrac{4}{3}\,nh^3\mathrm{E}\,(\lambda+\mu)^2 \\
&\qquad - \tfrac{4}{3}\,nh^3\mathrm{E}\,(\sigma_1+\sigma_2)(\lambda'+\mu') - \frac{16nh^3}{3a}\,\mathrm{E}\,(\lambda+\mu)(\sigma_1+\sigma_2) \quad . \quad . \quad . \quad . \quad . \qquad (14);
\end{aligned}$$

lastly

$$\tfrac{1}{2}n\int_{-h}^{h}\varpi'^2\,(1+h'/a)^2\,dh' = nh\,(1+\tfrac{1}{3}h^2/a^2)\,\varpi^2 + \tfrac{1}{3}\,nh^3p^2 + \tfrac{1}{3}nh^3\varpi p' + \frac{4nh^3}{3a}\,\varpi p \qquad (15).$$

Substituting from (12), (13), (14) and (15) in (11), the value of W per unit of area of the middle surface is,

$$\begin{aligned}
\mathrm{W} = 2nh&\left(1+\frac{h^2}{3a^2}\right)\{\sigma_1{}^2+\sigma_2{}^2+\mathrm{E}\,(\sigma_1+\sigma_2)^2+\tfrac{1}{2}\,\varpi^2\} \\
&+ \tfrac{2}{3}\,nh^3\,\{\lambda^2+\mu^2+\mathrm{E}\,(\lambda+\mu)^2+\tfrac{1}{2}\,p^2\} \\
&+ \tfrac{2}{3}\,nh^3\,(\mathfrak{A}\lambda'+\mathfrak{B}\mu'+\tfrac{1}{2}\,\varpi p') \\
&+ \frac{8nh^3}{3a}(\mathfrak{A}\lambda+\mathfrak{B}\mu+\tfrac{1}{2}\,\varpi p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16).
\end{aligned}$$

We must now obtain $\delta\mathfrak{T}$. We have

$$\delta\mathfrak{T} = \rho \iiint_{-h}^{h} (\ddot{u}'\,\delta u' + \ddot{v}'\,\delta v' + \ddot{w}'\,\delta w')\,(1 + h'/a)^2\, dh'\, d\mathrm{S}\,;$$

also

$$\int_{-h}^{h} \ddot{u}'\,\delta u'\,(1 + h'/a)^2\, dh' = 2h\left(1 + \frac{h^2}{3a^2}\right)\ddot{u}\,\delta u + \tfrac{2}{3}\,h^3\,\frac{d\ddot{u}}{dr}\,\frac{d\delta u}{dr} + \tfrac{1}{3}\,h^3\left(\ddot{u}\,\frac{d^2\delta u}{dr^2} + \frac{d^2\ddot{u}}{dr^2}\,\delta u\right)$$

$$+ \frac{4h^3}{3a}\left(\ddot{u}\,\frac{d\delta u}{dr} + \frac{d\ddot{u}}{dr}\,\delta u\right)$$

$$= 2h\left(1 + \frac{h^2}{3a^2}\right)\ddot{u}\,\delta u + \frac{2h^3}{3a^2}\left(\frac{d\ddot{w}}{d\theta} - \ddot{u}\right)\left(\frac{d\delta w}{d\theta} - \delta u\right)$$

$$+ \frac{h^3\mathrm{E}}{3a}\left(\ddot{u}\,\frac{d\delta\mathrm{K}}{d\theta} + \frac{d\ddot{\mathrm{K}}}{d\theta}\,\delta u\right) - \frac{4h^3}{3a^2}\left\{\ddot{u}\left(\frac{d\delta w}{d\theta} - \delta u\right) + \left(\frac{d\ddot{w}}{d\theta} - \ddot{u}\right)\delta u\right\}$$

by (2) and (3). Treating the other terms in a similar way, we obtain

$$\delta\mathfrak{T} = 2\rho h\,(1 + \tfrac{1}{3}\,h^2/a^2) \iint (\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w)\, d\mathrm{S}$$

$$+ \tfrac{2}{3}\,\rho h^3 \iint \left\{\frac{1}{a^2}\left(\frac{d\ddot{w}}{d\theta} - 3\ddot{u}\right)\left(\frac{d\delta w}{d\theta} - \delta u\right) + \frac{1}{a^2}\left(\frac{1}{\sin\theta}\,\frac{d\ddot{w}}{d\phi} - 3\ddot{v}\right)\left(\frac{1}{\sin\theta}\,\frac{d\delta w}{d\phi} - \delta v\right)\right.$$

$$\left. + \mathrm{E}\left(\mathrm{E}\ddot{\mathrm{K}} - \frac{2\ddot{w}}{a}\right)\delta\mathrm{K}\right\} d\mathrm{S}$$

$$+ \tfrac{1}{3}\,\rho h^3\mathrm{E} \iint \left\{\frac{\ddot{u}}{a}\,\frac{d\delta\mathrm{K}}{d\theta} + \frac{\ddot{v}}{a\sin\theta}\,\frac{d\delta\mathrm{K}}{d\phi} - \ddot{w}\,(\delta\lambda + \delta\mu) + \frac{1}{a}\,\frac{d\ddot{\mathrm{K}}}{d\theta}\,\delta u + \frac{1}{a\sin\theta}\,\frac{d\ddot{\mathrm{K}}}{d\phi}\,\delta v\right.$$

$$\left. - (\ddot{\lambda} + \ddot{\mu})\,\delta w\right\} d\mathrm{S}$$

$$- \frac{4\rho h^3}{3a} \iint \left\{\frac{1}{a}\left(\frac{d\ddot{w}}{d\theta} - \ddot{u}\right)\delta u + \frac{1}{a}\left(\frac{1}{\sin\theta}\,\frac{d\ddot{w}}{d\phi} - \ddot{v}\right)\delta v + \mathrm{E}\ddot{\mathrm{K}}\,\delta w\right\} d\mathrm{S} \quad . \quad . \quad . \quad . \quad (17).$$

We must next find $\delta\mathfrak{L}$.

We have

$$\delta\mathfrak{L} = \iint_{-h}^{h} (\mathrm{P}'\,\delta u' + \mathrm{U}'\,\delta v')\,(a + h')\sin\theta\, dh'\, d\phi + \iint_{-h}^{h} (\mathrm{Q}'\,\delta v' + \mathrm{U}'\,\delta u')\,(a + h')\, dh'\, d\theta$$

$$+ \int \mathrm{N}_2 a\sin\theta\,\delta w\, d\phi + \int \mathrm{N}_1 a\,\delta w\, d\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

whence

$$\delta\mathfrak{L} = \int\Big\{\mathrm{T}_1\,\delta u + \mathrm{M}_2\,\delta v + \mathrm{N}_2\,\delta w - \frac{\mathrm{G}_2}{a}\left(\frac{d\delta w}{d\theta} - \delta u\right) + \frac{\mathrm{H}_1}{a}\left(\frac{1}{\sin\theta}\,\frac{d\delta w}{d\phi} - \delta v\right)$$

$$+ \frac{2nh^3\mathrm{E}\mathfrak{A}}{3a}\,\frac{d\delta\mathrm{K}}{d\theta} + \frac{nh^3\mathrm{E}\varpi}{3a\sin\theta}\,\frac{d\delta\mathrm{K}}{d\phi}\Big\}a\sin\theta\,d\phi$$

$$+ \int\Big\{\mathrm{M}_1\,\delta u + \mathrm{T}_2\,\delta v + \mathrm{N}_1\,\delta w + \frac{\mathrm{G}_1}{a}\left(\frac{1}{\sin\theta}\,\frac{d\delta w}{d\phi} - \delta v\right) - \frac{\mathrm{H}_2}{a}\left(\frac{d\delta w}{d\theta} - \delta u\right)$$

$$+ \frac{2nh^3\mathrm{E}\mathfrak{B}}{3a\sin\theta}\,\frac{d\delta\mathrm{K}}{d\phi} + \frac{nh^3\mathrm{E}\varpi}{3a}\,\frac{d\delta\mathrm{K}}{d\theta}\Big\}a\,d\theta \quad . \quad . \quad . \quad (19).$$

Lastly,

$$\delta\mathrm{U} = \rho\iiint_{-h}^{h}(\mathrm{X}\,\delta u' + \mathrm{Y}\,\delta v' + \mathrm{Z}\,\delta w')(1 + h'/a)^2\,dh'd\mathrm{S}$$

$$= 2\rho h\left(1 + \frac{h^2}{3a^2}\right)\iint(\mathrm{X}\,\delta u + \mathrm{Y}\,\delta v + \mathrm{Z}\,\delta w)\,d\mathrm{S}$$

$$+ \tfrac{1}{3}\rho h^3\mathrm{E}\iint\left\{\frac{\mathrm{X}}{a}\,\frac{d\delta\mathrm{K}}{d\theta} + \frac{\mathrm{Y}}{a\sin\theta}\,\frac{d\delta\mathrm{K}}{d\phi} - \mathrm{Z}\,(\delta\lambda + \delta\mu)\right\}d\mathrm{S}$$

$$- \frac{4\rho h^3}{3a}\iint\left\{\frac{\mathrm{X}}{a}\left(\frac{d\delta w}{d\theta} - \delta u\right) + \frac{\mathrm{Y}}{a}\left(\frac{1}{\sin\theta}\,\frac{d\delta w}{d\phi} - \delta v\right) + \mathrm{ZE}\,\delta\mathrm{K}\right\}d\mathrm{S} \quad . \quad . \quad (20).$$

19. We shall, as in the case of a cylindrical shell, denote the four lines of W by W_1, W_2, W_3, W_4. Whence

$$\delta\mathrm{W}_1 = 4nh\,(1 + \tfrac{1}{3}h^2/a^2)\iint(\mathfrak{A}\delta\sigma_1 + \mathfrak{B}\delta\sigma_2 + \tfrac{1}{2}\varpi\delta\varpi)\,d\mathrm{S}$$

$$= 4nh\left(1 + \frac{h^2}{3a^2}\right)\left\{\int(\mathfrak{A}\delta u + \tfrac{1}{2}\varpi\,\delta v)\,a\sin\theta\,d\phi + \int(\mathfrak{B}\,\delta v + \tfrac{1}{2}\varpi\,\delta u)\,a\,d\theta\right\}$$

$$- 4nh\left(1 + \frac{h^2}{3a^2}\right)\iint\bigg[\left\{\frac{d}{d\theta}(\mathfrak{A}\sin\theta) - \mathfrak{B}\cos\theta + \tfrac{1}{2}\frac{d\varpi}{d\phi}\right\}\delta u$$

$$+ \left\{\frac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\,\frac{d}{d\theta}(\varpi\sin\theta) + \tfrac{1}{2}\varpi\cos\theta\right\}\delta v - (\mathfrak{A} + \mathfrak{B})\sin\theta\,\delta w\bigg]a\,d\theta\,d\phi \qquad (21),$$

from which we obtain the approximate equations

$$\left.\begin{aligned}\mathrm{T}_1 = 4nh\mathfrak{A}, \qquad \mathrm{T}_2 &= 4nh\mathfrak{B}\\ \mathrm{M}_1 = \mathrm{M}_2 &= 2nh\varpi\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

$$\left.\begin{aligned}\rho\ddot{u} &= \frac{2n}{a\sin\theta}\left\{\frac{d}{d\theta}(\mathfrak{A}\sin\theta) - \mathfrak{B}\cos\theta + \tfrac{1}{2}\frac{d\varpi}{d\phi}\right\} + \rho\mathrm{X}\\ \rho\ddot{v} &= \frac{2n}{a\sin\theta}\left\{\frac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\,\frac{d}{d\theta}(\varpi\sin\theta) + \tfrac{1}{2}\varpi\cos\theta\right\} + \rho\mathrm{Y}\\ \rho\ddot{w} &= -\frac{2n}{a}(\mathfrak{A} + \mathfrak{B}) + \rho\mathrm{Z}\end{aligned}\right\} \quad . \quad . \quad . \quad (23).$$

These are the equations which have been obtained by Mr. LOVE,[*] and which have been employed by him in discussing the extensional vibrations of a spherical shell.

Again

$$\delta \mathrm{W}_2 = \tfrac{4}{3} n h^3 \iint (\mathfrak{L} \delta\lambda + \mathfrak{F} \delta\mu + \tfrac{1}{2} p \delta p)\, d\mathrm{S}.$$

Substituting the values of λ, μ, p from (8) we obtain

$$\begin{aligned}\iint \mathfrak{L} \delta\lambda\, d\mathrm{S} &= -\iint \mathfrak{L} \left\{\frac{d^2\delta w}{d\theta^2} + \delta w + \mathrm{E}\left(\frac{d\delta u}{d\theta} + \frac{1}{\sin\theta}\frac{d\delta v}{d\phi} + \delta u \cot\theta + 2\delta w\right)\right\} \sin\theta\, d\theta\, d\phi \\ &= -\int \left\{\mathrm{E}\mathfrak{L} \sin\theta \delta u - \frac{d}{d\theta}(\mathfrak{L}\sin\theta)\,\delta w + \mathfrak{L}\sin\theta \frac{d\delta w}{d\theta}\right\} d\phi - \int \mathrm{E}\mathfrak{L}\delta v\, d\theta \\ &\quad + \iint \left[\mathrm{E}\sin\theta \frac{d\mathfrak{L}}{d\theta}\delta u + \mathrm{E}\frac{d\mathfrak{L}}{d\phi}\delta v \right. \\ &\qquad \left. - \left\{\frac{d^2}{d\theta^2}(\mathfrak{L}\sin\theta) + (1 + 2\mathrm{E})\,\mathfrak{L}\sin\theta\right\}\delta w\right] d\theta\, d\phi \quad (24),\end{aligned}$$

also

$$\begin{aligned}\iint \mathfrak{F}\delta\mu\, d\mathrm{S} &= -\iint \mathfrak{F} \left\{\frac{1}{\sin\theta}\frac{d^2\delta w}{d\phi^2} + \cos\theta \frac{d\delta w}{d\theta} + \delta w \sin\theta \right. \\ &\qquad \left. + \mathrm{E}\left(\frac{d\delta u}{d\theta}\sin\theta + \frac{d\delta v}{d\phi} + \delta u \cos\theta + 2\delta w \sin\theta\right)\right\} d\theta\, d\phi \\ &= -\int (\mathrm{E}\mathfrak{F}\sin\theta\delta u + \mathfrak{F}\cos\theta\delta w)\, d\phi \\ &\quad - \int \left(\mathrm{E}\mathfrak{F}\delta v - \frac{1}{\sin\theta}\frac{d\mathfrak{F}}{d\phi}\delta w + \frac{\mathfrak{F}}{\sin\theta}\frac{d\delta w}{d\phi}\right) d\theta \\ &\quad + \iint \left[\mathrm{E}\sin\theta \frac{d\mathfrak{F}}{d\theta}\delta u + \mathrm{E}\frac{d\mathfrak{F}}{d\phi}\delta v \right. \\ &\qquad \left. - \left\{\frac{1}{\sin\theta}\frac{d^2\mathfrak{F}}{d\phi^2} - \frac{d}{d\theta}(\mathfrak{F}\cos\theta) + (1 + 2\mathrm{E})\,\mathfrak{F}\sin\theta\right\}\delta w\right] d\theta\, d\phi \quad (25).\end{aligned}$$

In the last term $p\,\delta p$, we must treat the integral which involves $d^2\delta w/d\theta\, d\phi$ exactly in the same way as in the corresponding case of a cylindrical shell, and we shall thus obtain

$$\begin{aligned}\tfrac{1}{2}\iint p\,\delta p\, d\mathrm{S} &= \iint p\left(\cot\theta \frac{d\delta w}{d\phi} - \frac{d^2\delta w}{d\theta d\phi}\right) d\theta\, d\phi \\ &= \tfrac{1}{2}\int \left(\frac{dp}{d\phi}\delta w - p\frac{d\delta w}{d\phi}\right) d\phi + \int \left\{\left(p\cot\theta + \tfrac{1}{2}\frac{dp}{d\theta}\right)\delta w - \tfrac{1}{2}p\frac{d\delta w}{d\theta}\right\} d\theta \\ &\quad - \iint \left(\cot\theta \frac{dp}{d\phi} + \frac{d^2 p}{d\theta d\phi}\right)\delta w\, d\theta\, d\phi \quad \ldots\ldots\ldots\ldots \quad (26).\end{aligned}$$

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[*] 'Phil. Trans.,' A, 1888, p. 527. Equation (23) corresponds to LOVE's equations (46), (47), and (48) and (22) to (72).

Adding (24), (25), and (26), we finally obtain

$$\begin{aligned}\delta \mathrm{W}_2 = \frac{4nh^3}{3a}\int\Big[&-\{\mathfrak{E}+\mathrm{E}(\mathfrak{E}+\mathfrak{F})\}\,\delta u - \tfrac{1}{2}p\,\delta v + \frac{1}{\sin\theta}\left\{\frac{d}{d\theta}(\mathfrak{E}\sin\theta) - \mathfrak{F}\cos\theta\right.\\ &\left.+\tfrac{1}{2}\frac{dp}{d\phi}\right\}\delta w - \mathfrak{E}\left(\frac{d\delta w}{d\theta}-\delta u\right) - \tfrac{1}{2}p\left(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi}-\delta v\right)\Big]\,a\sin\theta\,d\phi\\ + \frac{4nh^3}{3a}\int\Big[&-\tfrac{1}{2}p\,\delta u - \{\mathfrak{F}+\mathrm{E}(\mathfrak{E}+\mathfrak{F})\}\,\delta v + \frac{1}{\sin\theta}\left(\frac{d\mathfrak{F}}{d\phi}+p\cos\theta+\tfrac{1}{2}\sin\theta\,\frac{dp}{d\theta}\right)\delta w\\ &-\mathfrak{F}\left(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi}-\delta v\right) - \tfrac{1}{2}p\left(\frac{d\delta w}{d\theta}-\delta u\right)\Big]\,a\,d\theta\\ + \tfrac{4}{3}nh^3\iint\Big[&\mathrm{E}\sin\theta\,\frac{d}{d\theta}(\mathfrak{E}+\mathfrak{F})\,\delta u + \mathrm{E}\,\frac{d}{d\phi}(\mathfrak{E}+\mathfrak{F})\,\delta v\\ &-\left\{\frac{d^2}{\delta\theta^2}(\mathfrak{E}\sin\theta) + (1+2\mathrm{E})(\mathfrak{E}+\mathfrak{F})\sin\theta + \frac{1}{\sin\theta}\frac{d^2\mathfrak{F}}{d\phi^2} - \frac{d}{d\theta}(\mathfrak{F}\cos\theta)\right.\\ &\left.+2(1+\mathrm{E})\,\mathfrak{F}\sin\theta + \cot\theta\,\frac{dp}{d\phi} + \frac{d^2p}{d\theta\,d\phi}\right\}\delta w\Big]\,d\theta\,d\phi \qquad (27).\end{aligned}$$

The expressions for W_3, W_4 may, as in the case of a cylindrical shell, be divided into two parts W_3', W_3'', W_4', W_4''. The values of $\delta\mathrm{W}_3'$, $\delta\mathrm{W}_4'$, may at once be written down from (21) by changing $\mathfrak{A}$, $\mathfrak{B}$, ϖ into $\mathfrak{E}'$, $\mathfrak{F}'$, p' and $\mathfrak{E}$, $\mathfrak{F}$, p respectively, and by altering the coefficient into $\frac{2}{3}nh^3$ and $8nh^3/3a$ respectively. With regard to W_3'' we have

$$\delta\mathrm{W}_3'' = \tfrac{2}{3}nh^3\iint(\mathfrak{A}\,\delta\lambda' + \mathfrak{B}\,\delta\mu' + \tfrac{1}{2}\varpi\,\delta p')\,d\mathrm{S}.$$

Substituting the value of λ' from (9) and integrating once by parts, we shall obtain

$$\begin{aligned}\iint\mathfrak{A}\,\delta\lambda'\,d\mathrm{S} = \mathrm{E}\int&\mathfrak{A}\,\frac{d\,\delta\mathrm{K}}{d\theta}\sin\theta\,d\phi\\ &-\iint\left[\mathrm{E}\,\frac{d}{d\theta}(\mathfrak{A}\sin\theta)\,\frac{d\,\delta\mathrm{K}}{d\theta} + \mathfrak{A}a\sin\theta\,\{2\,\delta\lambda + \mathrm{E}\,(\delta\lambda+\delta\mu)\}\right]d\theta\,d\phi.\end{aligned}$$

Treating the other terms in a similar manner, we shall finally obtain

$$\begin{aligned}\delta\mathrm{W}_3'' = &\int\left\{\frac{2nh^3\mathrm{E}\mathfrak{A}}{3a}\,\frac{d\,\delta\mathrm{K}}{d\theta} + \frac{nh^3\mathrm{E}\varpi}{3a\sin\theta}\,\frac{d\,\delta\mathrm{K}}{d\phi}\right\}a\sin\theta\,d\phi\\ &+\int\left\{\frac{2nh^3\mathrm{E}\mathfrak{B}}{3a\sin\theta}\,\frac{d\,\delta\mathrm{K}}{d\phi} + \frac{nh^3\mathrm{E}\varpi}{3a}\,\frac{d\,\delta\mathrm{K}}{d\theta}\right\}a\,d\theta\\ &-\frac{2nh^3}{3a^2}\iint\Big[\frac{\mathrm{E}}{\sin\theta}\left\{\frac{d}{d\theta}(\mathfrak{A}\sin\theta) - \mathfrak{B}\cos\theta + \tfrac{1}{2}\frac{d\varpi}{d\phi}\right\}\frac{d\delta\mathrm{K}}{d\theta}\\ &+\frac{\mathrm{E}}{\sin^2\theta}\left(\frac{d\mathfrak{B}}{d\phi} + \tfrac{1}{2}\frac{d\varpi}{d\theta}\sin\theta + \varpi\cos\theta\right)\frac{d\,\delta\mathrm{K}}{d\phi} + \mathrm{E}a\,(\mathfrak{A}+\mathfrak{B})\,(\delta\lambda+\delta\mu)\Big]\,d\mathrm{S}\\ &-\frac{4nh^3}{3a}\iint(\mathfrak{A}\,\delta\lambda + \mathfrak{B}\,\delta\mu + \tfrac{1}{2}\varpi\,\delta p)\,d\mathrm{S}.\end{aligned}$$

If in the first surface integral in this equation we substitute the approximate values of the coefficients of $d\,\delta K/d\theta$ &c. from (23), which we may do since this integral is multiplied by h^3, and then substitute the values of $\delta W_3''$, $\delta\mathfrak{T}$, $\delta\mathfrak{L}$, and δU in (10), it will be found that all the terms involving $d\,\delta K/d\theta$, $d\,\delta K/d\phi$, and $\delta\lambda + \delta\mu$ cut out; we are therefore no longer concerned with them, and the value of $\delta W_3''$ reduces to the last line; on this understanding we may write

$$\delta W_3'' + \delta W_4'' = \frac{4nh^3}{3a}\iint (\mathfrak{A}\,\delta\lambda + \mathfrak{B}\,\delta\mu + \tfrac{1}{2}\varpi\,\delta p)\,dS \quad . \quad . \quad . \quad (28).$$

The variation of the right-hand side of (28) might at once be written down from (27) by substituting $\mathfrak{A}$, $\mathfrak{B}$, and ϖ for $\mathfrak{E}$, $\mathfrak{F}$, and p; but it will be more convenient to present the results in another form. Taking the first term, and integrating the second differential coefficients *once* by parts, we obtain

$$\iint \mathfrak{A}\,\delta\lambda\,dS = -\int \mathfrak{A}\,\frac{d\,\delta w}{d\theta}\sin\theta\,d\phi + \iint\left\{\frac{d}{d\theta}(\mathfrak{A}\sin\theta)\frac{d\,\delta w}{d\theta} - \mathfrak{A}\sin\theta\,\delta w \right.$$
$$\left. - \mathrm{E}\mathfrak{A}a\sin\theta\,\delta K\right\} d\theta\,d\phi.$$

Treating the other terms in a similar way, and adding to the result from (21) that portion of δW_1 which depends upon h^3, and finally replacing the coefficients of $d\,\delta w/d\theta - \delta u$, &c., by their approximate values from (23), the final result will be

$$- \frac{4nh^3}{3a^2}\int\left\{\mathfrak{A}\left(\frac{d\,\delta w}{d\theta} - \delta u\right) + \tfrac{1}{2}\varpi\left(\frac{1}{\sin\theta}\frac{d\,\delta w}{d\phi} - \delta v\right)\right\} a\sin\theta\,d\phi$$
$$- \frac{4nh^3}{3a^2}\int\left\{\mathfrak{B}\left(\frac{1}{\sin\theta}\frac{d\,\delta w}{d\phi} - \delta v\right) + \tfrac{1}{2}\varpi\left(\frac{d\,\delta w}{d\theta} - \delta u\right)\right\} a\,d\theta$$
$$+ \frac{2\rho h^3}{3a^2}\iint\left\{(\ddot{u} - X)\left(\frac{d\,\delta w}{d\theta} - \delta u\right) + (\ddot{v} - Y)\left(\frac{1}{\sin\theta}\frac{d\,\delta w}{d\phi} - \delta v\right)\right.$$
$$\left. + \mathrm{E}a(\ddot{w} - Z)\,\delta K\right\} dS \quad (29).$$

This result enables us to test the accuracy of a portion of our work, and the fundamental hypothesis on which the theory is based; for if we substitute in (10) the expression (29), and also the value of $\delta\mathfrak{L}$ from (19), it will be seen that we have reproduced the values of the couples which are given by (7); also comparing with $\delta\mathfrak{L}$, the line integral parts of δW_2, given by (27), the line integral parts of $\delta W_3'$ and $\delta W_4'$, which, as we have explained above, are obtained from (21) by changing certain letters, we see that we have also reproduced the values of M_1, M_2, given by the third of (6). We may, therefore, omit the couple terms, and also the terms in M; also, since we have disposed of the terms in δW_1, which involve h^3, we shall write $\delta W_1'$ for the remaining portion which depends upon h, and the variational equation finally becomes

$$\begin{aligned}
&\delta W_1' + \delta W_2' + \delta W_3' + \delta W_4' + 2\rho h\,(1 + \tfrac{1}{3}\,h^2/a^2)\iint(\ddot{u}\,\delta u + \ddot{v}\,\delta v + \ddot{w}\,\delta w)\,dS \\
&\quad + \tfrac{2}{3}\,\rho h^3 \iint\left\{\frac{1}{a^2}\left(\frac{d\ddot{w}}{d\theta} - 2\ddot{u}\right)\left(\frac{d\,\delta w}{d\theta} - \delta u\right) + \frac{1}{a^2}\left(\frac{1}{\sin\theta}\frac{d\ddot{w}}{d\phi} - 2\ddot{v}\right)\left(\frac{1}{\sin\theta}\frac{d\,\delta w}{d\phi} - \delta v\right)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. + \mathrm{E}\left(\mathrm{E}\ddot{\mathrm{K}} - \frac{\ddot{w}}{a}\right)\delta\mathrm{K}\right\}dS \\
&\quad + \tfrac{1}{3}\,\rho h^3\mathrm{E}\iint\left\{\frac{1}{a}\frac{d\ddot{\mathrm{K}}}{d\theta}\,\delta u + \frac{1}{a\sin\theta}\frac{d\ddot{\mathrm{K}}}{d\phi}\,\delta v - (\ddot{\lambda} + \ddot{\mu})\,\delta w\right\}dS \\
&\quad - \frac{4\rho h^3}{3a}\iint\left\{\frac{1}{a}\left(\frac{d\ddot{w}}{d\theta} - \ddot{u}\right)\delta u + \frac{1}{a}\left(\frac{1}{\sin\theta}\frac{d\ddot{w}}{d\phi} - \ddot{v}\right)\delta v + \mathrm{E}\ddot{\mathrm{K}}\,\delta w\right\}dS \\
&= 2\rho h\left(1 + \frac{h^2}{3a^2}\right)\iint\Big(\mathrm{X}\,\delta u + \mathrm{Y}\,\delta v + \mathrm{Z}\,\delta w\Big)\,dS \\
&\quad - \frac{2\rho h^3}{3a^2}\iint\left\{\mathrm{X}\left(\frac{d\delta w}{d\theta} - \delta u\right) + \mathrm{Y}\left(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi} - \delta v\right) + \mathrm{ZE}a\,\delta\mathrm{K}\right\}dS \\
&\quad + \int(\mathrm{T}_1\,\delta u + \mathrm{N}_2\,\delta w)\,a\sin\theta\,d\phi + \int(\mathrm{T}_2\,\delta v + \mathrm{N}_1\,\delta w)\,a\,d\theta \quad \ldots\ldots \quad (30).
\end{aligned}$$

We have now got rid of all the terms involving the second differential coefficients of δu, δv, δw; and all that remains to be done is to integrate by parts the terms which involve the first differential coefficients. Putting

$$\alpha = \frac{d\ddot{w}}{d\theta} - 2\ddot{u} + \mathrm{X}, \qquad \beta = \frac{1}{\sin\theta}\frac{d\ddot{w}}{d\phi} - 2\ddot{v} + \mathrm{Y}, \qquad \gamma = \mathrm{E}\,(a\mathrm{E}\ddot{\mathrm{K}} - \ddot{w} + \mathrm{Z}) \quad (31),$$

we have

$$\begin{aligned}
&\frac{2\rho h^3}{3a^2}\iint\left\{\alpha\frac{d\delta w}{d\theta} + \frac{\beta}{\sin\theta}\frac{d\delta w}{d\phi} + a\gamma\delta\mathrm{K}\right\}dS \\
&\qquad\qquad = \frac{2\rho h^3}{3a}\int(\gamma\delta u + \alpha\delta w)\,a\sin\theta d\phi + \frac{2\rho h^3}{3a}\int(\gamma\delta v + \beta\delta w)\,ad\theta \\
&- \frac{2\rho h^3}{3a^2}\iint\left[\left\{\frac{d}{d\theta}(\gamma\sin\theta) - \gamma\cos\theta\right\}\delta u + \frac{d\gamma}{d\phi}\delta v + \left\{\frac{d}{d\theta}(\alpha\sin\theta) + \frac{d\beta}{d\phi} - 2\gamma\sin\theta\right\}\delta w\right]\frac{dS}{\sin\theta} \quad (32).
\end{aligned}$$

Substituting the values of $\delta W_1'$, $\delta W_2'$, $\delta W_3'$, $\delta W_4'$, and the right hand side of (32) in (30), and picking out the line integral terms, we obtain the following equations for the sectional stresses, viz.,

$$\left.\begin{aligned}
\mathrm{T}_1 &= 4nh\mathfrak{A} + \frac{4nh^3}{3a}\{\mathfrak{E} - \mathrm{E}(\mathfrak{E}+\mathfrak{F})\} + \tfrac{2}{3}nh^3\mathfrak{E}' + \frac{2\rho h^3}{3a}\mathrm{E}(a\mathrm{E}\ddot{\mathrm{K}} - \ddot{w} + z)\\
\mathrm{M}_2 &= 2nh\varpi + \frac{2nh^3}{3a}p + \tfrac{1}{3}nh^3p'\\
\mathrm{N}_2 &= \frac{4nh^3}{3a\sin\theta}\left\{\frac{d}{d\theta}(\mathfrak{E}\sin\theta) - \mathfrak{F}\cos\theta + \tfrac{1}{2}\frac{dp}{d\phi}\right\} + \frac{2\rho h^3}{3a}\left(\frac{d\ddot{w}}{d\theta} - 2\ddot{u} + \mathrm{X}\right)\\
\mathrm{G}_2 &= \tfrac{4}{3}nh^3(\mathfrak{E} + \mathfrak{A}/a)\\
\mathrm{H}_1 &= -\tfrac{2}{3}nh^3(p + \varpi/a)
\end{aligned}\right\} \quad (33)$$

which give the values of the sectional stresses across a parallel of latitude; and

$$\left.\begin{aligned}
\mathrm{M}_1 &= 2nh\varpi + \frac{2nh^3}{3a}p + \tfrac{1}{3}nh^3p'\\
\mathrm{T}_2 &= 4nh\mathfrak{B} + \frac{4nh^3}{3a}\{\mathfrak{F} - \mathrm{E}(\mathfrak{E}+\mathfrak{F})\} + \tfrac{2}{3}nh^3\mathfrak{F}' + \frac{2\rho h^3}{3a}\mathrm{E}(a\mathrm{E}\ddot{\mathrm{K}} - \ddot{w} + \mathrm{Z})\\
\mathrm{N}_1 &= \frac{4nh^3}{3a\sin\theta}\left(\frac{d\mathfrak{F}}{d\phi} + p\cos\theta + \tfrac{1}{2}\sin\theta\frac{dp}{d\theta}\right) + \frac{2\rho h^3}{3a}\left(\frac{1}{\sin\theta}\frac{d\ddot{w}}{d\phi} - 2\ddot{v} + \mathrm{Y}\right)\\
\mathrm{G}_1 &= -\tfrac{4}{3}nh^3(\mathfrak{F} + \mathfrak{B}/a)\\
\mathrm{H}_2 &= \tfrac{2}{3}nh^3(p + \varpi/a)
\end{aligned}\right\} \quad (34)$$

which give the values of the sectional stresses across a meridian.

In these equations we may, if we please, substitute the approximate values of $\ddot{u}$, $\ddot{v}$, $\ddot{w}$ from (23), and by means of these values it can be shown that the values of N_1, N_2 agree with the values which are obtained by substituting the values of the couples in the fourth and fifth of (5).

Picking out the coefficients of δu, δv, δw, in the surface integrals, we obtain the equations of motion, which are

$$\begin{aligned}
&\left\{2\left(1 + \frac{h^2}{a^2}\right)\ddot{u} + \frac{h^2\mathrm{E}}{3a}\frac{d\ddot{\mathrm{K}}}{d\theta} - \frac{4h^2}{3a^2}\frac{d\ddot{w}}{d\theta} - 2\left(1 + \frac{h^2}{3a^2}\right)\mathrm{X}\right\}\rho a\sin\theta\\
&\quad = 4n\left\{\frac{d}{d\theta}(\mathfrak{A}\sin\theta) - \mathfrak{B}\cos\theta + \tfrac{1}{2}\frac{d\varpi}{d\phi}\right\}\\
&\qquad - \frac{4nh^2}{3a^2}\mathrm{E}\sin\theta\frac{d}{d\theta}(\mathfrak{E} + \mathfrak{F}) + \tfrac{2}{3}nh^2\left\{\frac{d}{d\theta}(\mathfrak{E}'\sin\theta) - \mathfrak{F}'\cos\theta + \tfrac{1}{2}\frac{dp'}{d\phi}\right\}\\
&\qquad + \frac{8nh^2}{3a}\left\{\frac{d}{d\theta}(\mathfrak{E}\sin\theta) - \mathfrak{F}\cos\theta + \tfrac{1}{2}\frac{dp}{d\phi}\right\} + \frac{2\rho h^2}{3a}\left\{\alpha\sin\theta + \frac{d\gamma}{d\theta}\sin\theta - \gamma\cos\theta\right\} \quad (35),
\end{aligned}$$

$$\left\{2\left(1+\frac{h^2}{a^2}\right)\ddot{v}+\frac{h^2\mathrm{E}}{3a\sin\theta}\frac{d\ddot{\mathrm{K}}}{d\phi}-\frac{4h^2}{3a^2\sin\theta}\frac{d\ddot{w}}{d\phi}-2\left(1+\frac{h^2}{3a^2}\right)\mathrm{Y}\right\}\rho a\sin\theta$$
$$=4n\left\{\frac{d\mathfrak{B}}{d\phi}+\tfrac{1}{2}\frac{d}{d\theta}(\varpi\sin\theta)+\tfrac{1}{2}\varpi\cos\theta\right\}$$
$$-\frac{4nh^2}{3a}\mathrm{E}\frac{d}{d\phi}(\mathfrak{E}+\mathfrak{F})+\tfrac{2}{3}nh^2\left\{\frac{d\mathfrak{F}'}{d\phi}+\tfrac{1}{2}\frac{d}{d\theta}(p'\sin\theta)+\tfrac{1}{2}p'\cos\theta\right\}$$
$$+\frac{8nh^2}{3a}\left\{\frac{d\mathfrak{F}}{d\phi}+\tfrac{1}{2}\frac{d}{d\theta}(p\sin\theta)+\tfrac{1}{2}p\cos\theta\right\}+\frac{2\rho h^2}{3a}\left(\beta\sin\theta+\frac{d\gamma}{d\phi}\right)\quad . \quad . \quad (36).$$

$$\left\{2\left(1+\frac{h^2}{3a^2}\right)(\ddot{w}-\mathrm{Z})-\tfrac{1}{3}h^2\mathrm{E}(\ddot{\lambda}+\ddot{\mu})-\frac{4h^2}{3a}\mathrm{E}\ddot{\mathrm{K}}\right\}\rho a\sin\theta$$
$$=-4n(\mathfrak{A}+\mathfrak{B})\sin\theta+\frac{4nh^2}{3a}\left\{\frac{d^2}{d\theta^2}(\mathfrak{E}\sin\theta)+(1+2\mathrm{E})(\mathfrak{E}+\mathfrak{F})\sin\theta\right.$$
$$\left.+\frac{1}{\sin\theta}\frac{d^2\mathfrak{F}}{d\phi^2}-\frac{d}{d\theta}(\mathfrak{F}\cos\theta)+\cot\theta\frac{dp}{d\phi}+\frac{d^2p}{d\theta\,d\phi}\right\}$$
$$-\tfrac{2}{3}nh^2(\mathfrak{E}'+\mathfrak{F}')\sin\theta-\frac{8nh^2}{3a}(\mathfrak{E}+\mathfrak{F})\sin\theta$$
$$+\frac{2\rho h^2}{3a}\left\{\frac{d}{d\theta}(\alpha\sin\theta)+\frac{d\beta}{d\phi}-2\gamma\sin\theta\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37).$$

The correctness of these equations may be tested by substituting the values of the sectional stresses from (33) and (34) in the first three of (5), when it will be found that we shall reproduce (35), (36), and (37).

20. The boundary conditions for a spherical shell may be investigated in exactly the same manner as in the case of a cylindrical shell, by means of STOKES' theorem; for in the present case the theorem may be written

$$\int\left(\frac{d\mathrm{H}'}{d\phi}\delta w+\mathrm{H}'\frac{d\delta w}{d\phi}\right)d\phi+\int\left(\frac{d\mathrm{H}'}{d\theta}\delta w+\mathrm{H}'\frac{d\delta w}{d\theta}\right)d\theta=0,$$

the integration extending round any curvilinear rectangle bounded by two meridians and two parallels of latitude. If, therefore, in the figure we apply to the side AD the stresses,

$$\mathrm{M}_2'=\mathrm{H}'/a,\qquad \mathrm{N}_2'=\frac{1}{a\sin\theta}\frac{d\mathrm{H}'}{d\phi},\qquad \mathrm{H}_1'=\mathrm{H}';$$

to the side BD the stresses

$$\mathrm{M}_1'=\mathrm{H}'/a,\qquad \mathrm{N}_1'=\frac{1}{a}\frac{d\mathrm{H}'}{d\theta},\qquad \mathrm{H}_2'=-\mathrm{H}';$$

and to the sides OB, OA, corresponding and opposite stresses respectively, the preceding integral becomes

$$\int\left\{\mathrm{M}_2'\,\delta v + \mathrm{N}_2'\,\delta w + \frac{\mathrm{H}_1'}{a}\left(\frac{1}{\sin\theta}\frac{d\delta w}{d\phi} - \delta v\right)\right\} a\sin\theta\,d\phi$$
$$+ \int\left\{\mathrm{M}_1'\,\delta u + \mathrm{N}_1'\,\delta w - \frac{\mathrm{H}_2'}{a}\left(\frac{d\delta w}{d\theta} - \delta u\right)\right\} a\,d\theta = 0\,,$$

which shows that the work done by this system of stresses is zero.

If, therefore, we suppose that the rectangle OADB, instead of being under the action of the remainder of the shell, is isolated, and that its state of strain is maintained by stresses applied to its edges, then it follows that if instead of the torsional couples H_1, H_2, due to the action of contiguous portions of the shell, we apply torsional couples $\mathfrak{H}_1$, $\mathfrak{H}_2$, where

$$\mathfrak{H}_1 = \mathrm{H}_1 + \mathrm{H}' \quad \text{. . . . .} \quad (38),$$

$$\mathfrak{H}_2 = \mathrm{H}_2 - \mathrm{H}' \quad \text{. . . . .} \quad (39),$$

the state of strain will remain unchanged, provided we apply in addition the stresses

$$\left.\begin{aligned} \mathfrak{M}_2 &= \mathrm{M}_2 + \mathrm{H}'/a \\ \mathfrak{N}_2 &= \mathrm{N}_2 + \frac{1}{a\sin\theta}\frac{d\mathrm{H}'}{d\phi} \end{aligned}\right\} \quad \text{. . . . .} \quad (40)$$

and

$$\left.\begin{aligned} \mathfrak{M}_1 &= \mathrm{M}_1 + \mathrm{H}'/a \\ \mathfrak{N}_1 &= \mathrm{N}_1 + \frac{1}{a}\frac{d\mathrm{H}'}{d\theta} \end{aligned}\right\} \quad \text{. . . . .} \quad (41),$$

whence eliminating H′ between (38) and (40), and between (39) and (41) respectively, we obtain

$$\left.\begin{aligned} \mathfrak{M}_2 a - \mathfrak{H}_1 &= \mathrm{M}_2 a - \mathrm{H}_1 \\ \mathfrak{N}_2 a\sin\theta - \frac{d\mathfrak{H}_1}{d\phi} &= \mathrm{N}_2 a\sin\theta - \frac{d\mathrm{H}_1}{d\phi} \end{aligned}\right\} \quad \text{. . . . . . .} \quad (42)$$

and

$$\left.\begin{aligned} \mathfrak{M}_1 a + \mathfrak{H}_2 &= \mathrm{M}_1 a + \mathrm{H}_2 \\ \mathfrak{N}_1 a + \frac{d\mathfrak{H}_2}{d\theta} &= \mathrm{N}_2 a + \frac{d\mathrm{H}_2}{d\theta} \end{aligned}\right\} \quad \text{. . . . .} \quad (43)$$

in which we are to remember that $\mathfrak{H}_1 = -\mathfrak{H}_2$, and $\mathrm{H}_1 = -\mathrm{H}_2$.

In these equations the Roman letters denote the stresses due to the action of contiguous portions of the shell, whilst the Old English letters denote the values of the actual stresses applied to the boundary. If, therefore, the shell consists of a portion of a sphere bounded by two meridians and two parallels of latitude, and whose edges are free, the boundary conditions along a parallel of latitude are obtained by equating the right hand sides of the first and fourth of (33) and of (42) to zero; whilst the boundary conditions along a meridian are similarly obtained by equating the right hand sides of the first and fourth of (34) and of (43) to zero.

21. If the shell is supposed to vibrate in such a manner, that its middle surface does not experience any extension or contraction throughout the motion, the equations of motion can be obtained by taking the variation subject to the conditions of inextensibility, and introducing indeterminate multipliers.

22. It will now be convenient to make a short digression for the purpose of considering some of the quantities involved.

Let P be any point on the deformed middle surface whose undisplaced coordinates are (a, θ, ϕ). The coordinates of P after deformation are

$$\mathrm{R} = a + w, \qquad \Theta = \theta + u/a, \qquad \Phi = \phi + v/a \sin\theta \quad . \quad . \quad . \quad (44),$$

and since u, v, w are functions of θ and ϕ, the elimination of the latter quantities from (44) will give a relation between R, Θ, Φ, which is the equation of the deformed middle surface.

If ρ_1 be the radius of curvature at any point of a meridian section after deformation, and P the perpendicular from the centre on to the tangent at that point to the deformed section,

$$\frac{1}{\rho_1} = \frac{1}{\mathrm{R}}\frac{d\mathrm{P}}{d\mathrm{R}}.$$

Now

$$\frac{1}{\mathrm{P}^2} = \frac{1}{\mathrm{R}^2}\left\{1 + \left(\frac{d\mathrm{R}}{\mathrm{R}d\Theta}\right)^2\right\}$$

$$= \frac{1}{\mathrm{R}^2}\left\{1 + \frac{(dw/d\theta)^2}{\mathrm{R}^2(1 + du/ad\theta)^2}\right\},$$

and therefore, neglecting cubes of displacements,

$$\mathrm{P} = a + w - \frac{1}{2a}\left(\frac{dw}{d\theta}\right)^2.$$

Also

$$d\mathrm{R} = \frac{dw}{d\theta}\,d\theta,$$

whence

$$\frac{1}{\rho_1} - \frac{1}{a} = -\frac{1}{a^2}\left(\frac{d^2w}{d\theta^2} + w\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45),$$

which gives the change of curvature along a meridian.

We shall now find an expression for the change of curvature along any great circle which makes an angle γ with a meridian.

In the figure on p. 463 join OD, and let the angle OED $= \chi$, and the angle DOA $= \gamma$; then by (45) the change of curvature along OD is

$$-\frac{1}{a^2}\left(\frac{d^2w}{d\chi^2}+w\right).$$

If $w+\delta w$ be the normal displacement at D, it follows by equating the two values of δw that

$$\frac{dw}{d\chi}\delta\chi+\tfrac{1}{2}\frac{d^2w}{d\chi^2}\delta\chi^2=\frac{dw}{d\theta}\delta\theta+\frac{dw}{d\phi}\delta\phi+\tfrac{1}{2}\left(\frac{d^2w}{d\theta^2}\delta\theta^2+2\frac{d^2w}{d\theta d\phi}\delta\theta\,\delta\phi+\frac{d^2w}{d\phi^2}\delta\phi^2\right) \quad (46).$$

From the spherical triangle ODP we have

$$-\cos\gamma=\frac{\cos \mathrm{PD}-\cos\theta\cos\delta\chi}{\sin\theta\sin\delta\chi},$$

whence

$$\begin{aligned}\sin\theta\,\delta\theta&=\cos\gamma\sin\theta\,\delta\chi+\tfrac{1}{2}\cos\theta\,\delta\chi^2-\tfrac{1}{2}\cos\theta\,\delta\theta^2\\&=\cos\gamma\sin\theta\,\delta\chi+\tfrac{1}{2}\cos\theta\sin^2\gamma\,\delta\chi^2.\end{aligned}$$

Again

$$\frac{\sin\delta\phi}{\sin\delta\chi}=\frac{\sin\gamma}{\sin(\theta+\delta\theta)},$$

whence

$$\delta\phi=\frac{\sin\gamma}{\sin\theta}(\delta\chi-\cot\theta\cos\gamma\,\delta\chi^2).$$

Substituting these values of $\delta\theta$, $\delta\phi$ in (46) and equating coefficients of $\delta\chi^2$, we obtain

$$\frac{d^2w}{d\chi^2}=\frac{\sin^2\gamma}{\sin^2\theta}\frac{d^2w}{d\phi^2}+\frac{\sin 2\gamma}{\sin\theta}\frac{d^2w}{d\theta d\phi}+\cos^2\gamma\frac{d^2w}{d\theta^2}-\frac{\sin 2\gamma\cos\theta}{\sin^2\theta}\frac{dw}{d\phi}+\sin^2\gamma\cot\theta\frac{dw}{d\theta} \quad (47).$$

Whence it follows, that if ρ_1, ρ_2 are the principal radii of curvature along and perpendicular to a meridian

$$\left.\begin{aligned}\frac{1}{\rho_1}-\frac{1}{a}&=-\frac{1}{a^2}\left(\frac{d^2w}{d\theta^2}+w\right)\\\frac{1}{\rho_2}-\frac{1}{a}&=-\frac{1}{a^2}\left(\frac{1}{\sin^2\theta}\frac{d^2w}{d\phi^2}+\cot\theta\frac{dw}{d\theta}+w\right)\end{aligned}\right\}\quad .\ .\ .\ .\ (48).$$

23. When the middle surface is inextensible, it has been shown by Lord RAYLEIGH* that the displacements are given by the equations

$$\left.\begin{aligned} u &= -\Sigma \mathrm{A}_s\, \epsilon^{\iota s\phi} \sin\theta \tan^s \tfrac{1}{2}\theta \\ v &= \Sigma \iota \mathrm{A}_s\, \epsilon^{\iota s\phi} \sin\theta \tan^s \tfrac{1}{2}\theta \\ w &= \Sigma \mathrm{A}_s\, \epsilon^{\iota s\phi} (s + \cos\theta) \tan^s \tfrac{1}{2}\theta \end{aligned}\right\} \quad \ldots\ldots \quad (49).$$

where $s = 2, 3, 4 \ldots$ and A_s is a complex function of the time. From these equations it can easily be shown by means of (48) that

$$\frac{1}{\rho_1} - \frac{1}{a} = -\Sigma \frac{\mathrm{A}_s (s^3 - s)\, \epsilon^{\iota s\phi} \tan^s \frac{1}{2}\theta}{a \sin^2\theta} = -\left(\frac{1}{\rho_2} - \frac{1}{a}\right) \quad \ldots \quad (50).$$

The value of the potential energy is given by the second line of (16); also by the first two of (8) and by (48)

$$\lambda = -\mu = \frac{1}{\rho_1} - \frac{1}{a} \quad \ldots\ldots \quad (51),$$

and by the last of (8)

$$p = -2\iota \Sigma \frac{\mathrm{A}_s (s^3 - s)\, \epsilon^{\iota s\phi} \tan^s \frac{1}{2}\theta}{a \sin^2\theta},$$

whence

$$\mathrm{W} = \frac{4nh^3}{3a^2 \sin^4\theta} \left[\{\Sigma\, \mathrm{A}_s (s^3 - s) \cos s\phi \tan^s \tfrac{1}{2}\theta\}^2 + \{\Sigma\, \mathrm{A}_s (s^3 - s) \sin s\phi \tan^s \tfrac{1}{2}\theta\}^2\right] \quad (52),$$

which agrees with Lord RAYLEIGH's result.

Let us now suppose that a bell which consists of a spherical shell, bounded by a small circle whose latitude is $\frac{1}{2}\pi - \alpha$, is vibrating in such a manner that its middle surface does not undergo any extension or contraction throughout the motion. One of the boundary conditions requires that the flexural couple G_2 should vanish along the circle of latitude which constitutes the free edge of the bell. By (7) and (51)

$$\begin{aligned} \mathrm{G}_2 &= \tfrac{4}{3}\, nh^3 \mathfrak{E} = \tfrac{4}{3}\, nh^3 \{\lambda + \mathrm{E}(\lambda + \mu)\} \\ &= \tfrac{4}{3}\, nh^3 \left(\frac{1}{\rho_1} - \frac{1}{a}\right). \end{aligned}$$

From (50) we see that G_2 cannot vanish for any value of θ except $\theta = 0$, that is, at the pole, provided $s > 2$. It, therefore, follows that a spherical bell whose edge is free cannot vibrate in this manner if the middle surface is supposed to remain absolutely inextensible throughout the motion. If, however, extension or contraction were to take place in the neighbourhood of the edge, it would be possible for G_2 to vanish there, and also to satisfy the other boundary conditions.

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* 'London Math. Soc. Proc.,' vol 13, p. 4 (1881).

There seems no reason to doubt that the argument which has been employed in the case of a cylindrical shell, would apply equally to the case of a spherical shell, and probably also to a shell of any shape; in which case, the portions of the displacements upon which extension principally depends, would be small compared with the portions upon which bending principally depends, except at points whose distances from a free edge are comparable with the thickness. At the same time it would be very desirable to obtain the solution of some problem relating to the vibrations of a shell whose edges are free, in which no supposition is made as to the relative magnitudes of the extensional and flexural terms.